

Clairaut anti-invariant submersions from Lorentzian trans-Sasakian manifolds

Clairaut anti-invariant submersions

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Abstract

Purpose – The central idea of this research article is to examine the characteristics of Clairaut submersions from Lorentzian trans-Sasakian manifolds of type (α, β) and also, to enhance this geometrical analysis with some specific cases, namely Clairaut submersion from Lorentzian α -Sasakian manifold, Lorentzian β -Kenmotsu manifold and Lorentzian cosymplectic manifold. Furthermore, the authors discuss some results about Clairaut Lagrangian submersions whose total space is a Lorentzian trans-Sasakian manifolds of type (α, β) . Finally, the authors furnished some examples based on this study.

Design/methodology/approach – This research discourse based on classifications of submersion, mainly Clairaut submersions, whose total manifolds is Lorentzian trans-Sasakian manifolds and its all classes like Lorentzian Sasakian, Lorentzian Kenmotsu and Lorentzian cosymplectic manifolds. In addition, the authors have explored some axioms of Clairaut Lorentzian submersions and illustrates our findings with some non-trivial examples.

Findings – The major finding of this study is to exhibit a necessary and sufficient condition for a submersions to be a Clairaut submersions and also find a condition for Clairaut Lagrangian submersions from Lorentzian trans-Sasakian manifolds.

Originality/value – The results and examples of the present manuscript are original. In addition, more general results with fair value and supportive examples are provided.

Keywords Clairaut submersion, Anti-invariant submersion, Lorentzian trans-Sasakian manifolds, Clairaut Lagrangian submersion

Paper type Research paper

1. Introduction

The conception of Riemannian immersion is studied extensively together with starting the study of Riemannian geometry. In fact, Riemannian manifolds are studied first as surfaces imbedded in \mathbb{R}^3 . In 1956, Nash [1] proved that a revolution for Riemannian manifold that all

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Riemannian manifolds are isometrically embedded at any small part of Euclidean space. Consequently, the differential geometry of the Riemannian immersion is commonly noted, and it can be found in different text books such as ([2, 3]).

Contrastingly, “dual” concept of Riemannian immersions is one of the famous research fields in differential geometry and is the theory of Riemannian submersions, which was first investigated by O’Neill [4] and Gray [5]. Watson [6] popularized the knowledge of Riemannian submersions considering almost Hermitian manifolds in terms of almost Hermitian submersions. Afterward, almost Hermitian submersions are discussed with in various subcategories of almost Hermitian manifolds. Also, Riemannian submersions are enhanced considering many subcategories of almost contact metric manifolds in terms of contact Riemannian submersions. Several materials about Riemannian, almost Hermitian or contact Riemannian submersions are available in reference [2].

Most of the research linked to the theory of anti-invariant Riemannian, Lagrangian submersions and Clairaut anti-invariant submersions is available in Şahin’s book [3]. Afterward, several kinds of Riemannian submersions appeared, for example: semi-invariant, slant, pointwise-slant, semi-slant, hemi-slant and generic submersions. Most of the studies related to these can also be found in Şahin’s book [3].

In 1972, Bishop [7] proposed the concept and conditions of a Clairaut submersion in terms of a natural generalization of a surface of revolution. Under these circumstances, for every geodesic σ at the surface \mathcal{S} , function $\gamma \sin \Theta$ is constant through σ , here γ is a metric between the point at surface and rotation axis, also Θ defines angle within σ and meridian through σ .

The concept of anti-invariant Riemannian and Clairaut anti-invariant submersion has been fitting a very progressive geometric analysis field since Şahin [8] essentially described such submersions of almost Hermitian manifolds on Riemannian manifolds. Indeed, anti-invariant Riemannian and Clairaut anti-invariant submersion have been examined in various types of geometrical manifolds, namely Kähler [8–10], almost product [11], Sasakian [12, 13], Kenmotsu [13], cosymplectic [30], paracosymplectic [14, 15] and trans-Sasakian manifolds [16–18]. Note that this concept of anti-invariant Riemannian submersion is generalized to conformal anti-invariant submersions [19–21].

In [22], Allison proposed Clairaut submersions in case the total manifold is Lorentzian. In addition, it is discovered that Clairaut submersions are used for static spacetime applications. Basically, a static spacetime can be considered as a Lorentzian manifold.

On the other hand, in 2013, De *et al.* [23] presented the concept of Lorentzian trans-Sasakian manifolds. Trans-Sasakian structure together with Lorentzian metric can be applied naturally at the odd dimensional manifold. Motivated by above research studies mentioned in this paper, we have examined the Clairaut anti-invariant submersions from Lorentzian trans-Sasakian manifolds.

The work is ordered as follows. Section 2 presents basic notion and definition for Lorentzian trans-Sasakian manifolds. Section 3 includes particular background of Riemannian submersions. Section 4 presents definition of anti-invariant and Lagrangian submersions. In section 5, we study anti-invariant submersions and Clairaut anti-invariant submersion from trans-Sasakian manifolds onto Riemannian manifolds admitting horizontal Reeb vector field. In section 6, we deal with some axioms of Clairaut Lagrangian submersion and provide some examples and some of their characteristic properties.

2. Lorentzian trans-Sasakian manifolds

A $(2n + 1)$ -dimensional differentiable manifold M is named the *Lorentzian Trans-Sasakian manifold* [23] in case it allows $(1, 1)$ tensor field φ , the global vector field ζ named *Reeb vector*

field or contra-variant vector field, that is, in case η is a dual 1-form of ζ , and the Lorentzian metric g that satisfies [24].

$$\varphi^2 U = U + \eta(U)\zeta, \quad \eta(\zeta) = -1, \quad \varphi\zeta = 0, \quad (2.1)$$

$$g(\varphi U, \varphi V) = g(U, V) + \eta(U)\eta(V), \quad (2.2)$$

where both U and V refer to any vector fields at M . Also, using previous axioms gives

$$\eta \circ \varphi = 0 \quad \text{and} \quad \eta(U) = g(U, \zeta). \quad (2.3)$$

Here, $(\varphi, \zeta, \eta, g)$ [23] is named *Lorentzian structure* of M . A *Lorentzian trans-Sasakian manifold* M also satisfies [25].

$$(\mathcal{D}_U \varphi)V = \alpha[g(U, V)\zeta - \eta(V)U] + \beta[g(\varphi U, V)\zeta - \eta(V)\varphi U] \quad (2.4)$$

for functions α and β and \mathcal{D} is Levi-Civita connection with respect to the Lorentzian metric g at M . Moreover, $(M, \varphi, \zeta, \eta, g)$ is named the Lorentzian trans-Sasakian manifold from type (α, β) ; for more details, see ([26]). It can be deduced from (2.1) that

$$\mathcal{D}_U \zeta = -\alpha\varphi U - \beta(U + \eta(U)\zeta). \quad (2.5)$$

Remark 1.

- (1) If $\alpha = 0$ and $\beta \neq 0$ (or $\beta = 1$), therefore the manifold turns into the *Lorentzian β -Kenmotsu manifold (or Lorentzian Kenmotsu manifold)* [23].
- (2) If $\alpha \neq 0$ (or $\alpha = 1$) and $\beta = 0$, therefore this manifold turns into the *Lorentzian α -Sasakian manifold (or Lorentzian Sasakian manifold)* [23].
- (3) In case $\alpha = 0$ and β , therefore, the manifold turns into the *Lorentzian cosymplectic manifold* [23].

3. Riemannian submersions

An essential background of Riemannian submersions is given at this part.

Suppose (M, g) and (N, g_N) are Riemannian manifolds, such that $\dim(M) > \dim(N)$. The subjective mapping $\psi: (M, g) \rightarrow (N, g_N)$ is named the *Riemannian submersion* [4] if:

(S1) The $\text{rank}(\psi) = \dim(N)$.

Therefore, for all $q \in N$, $\psi^{-1}(q) = \psi_q^{-1}$ is the k -dimensional submanifold of M and is named the *fiber*, with

$$k = \dim(M) - \dim(N).$$

The vector field at M is named *vertical* (resp. *horizontal*) in case it is still as a tangent (orthogonal) relating to the fibers. The vector field X at M is named *basic* in case X is horizontal and ψ -connecting to the vector field X_* at N , which means $\psi_*(X_p) = X_{*\psi(p)}$ for any $p \in M$, where ψ_* is derivative or differential map of ψ . \mathcal{V} and \mathcal{H} define the projections at vertical distribution $\ker\psi_*$ and horizontal distribution $\ker\psi_*^\perp$, in the same order. Usually, a manifold (M, g) is named the *total manifold* and (N, g_N) is named *base manifold* of the submersion $\psi: (M, g) \rightarrow (N, g_N)$.

(S2) ψ_* preserves the lengths of horizontal vectors.

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This condition is equivalent to say that the derivative map ψ_* of ψ , restricted to $\ker\psi_*^\perp$, is the linear isometry. The geometrical description of Riemannian submersions is represented by O'Neill's tensors \mathcal{T} and \mathcal{A} , determined as:

$$\mathcal{T}_{E_1}F_1 = \mathcal{V}\mathcal{D}_{\mathcal{V}E_1}\mathcal{H}F_1 + \mathcal{H}\mathcal{D}_{\mathcal{V}E_1}\mathcal{V}F_1, \tag{3.1}$$

$$\mathcal{A}_{E_1}F_1 = \mathcal{V}\mathcal{D}_{\mathcal{H}E_1}\mathcal{H}F_1 + \mathcal{H}\mathcal{D}_{\mathcal{H}E_1}\mathcal{V}F_1 \tag{3.2}$$

for any vector fields E_1 and F_1 at M , with \mathcal{D} is Levi-Civita connection of g . Clearly, \mathcal{T}_{E_1} in addition to \mathcal{A}_{E_1} are skew-symmetric operators at tangent bundle of M reversing vertical and the horizontal distributions. To sum up, tensor fields properties \mathcal{T} as well as \mathcal{A} , Suppose V_1, W_1 are vertical and X_1, Y_1 are horizontal vector fields at M , therefore

$$\mathcal{T}_{V_1}W_1 = \mathcal{T}_{W_1}V_1, \tag{3.3}$$

$$\mathcal{A}_{X_1}Y_1 = -\mathcal{A}_{Y_1}X_1 = \frac{1}{2}\mathcal{V}[X_1, Y_1]. \tag{3.4}$$

On the other hand, from (3.1) and (3.2), we obtain

$$\mathcal{D}_{V_1}W_1 = \mathcal{T}_{V_1}W_1 + \widehat{\mathcal{D}}_{V_1}W_1, \tag{3.5}$$

$$\mathcal{D}_{V_1}X_1 = \mathcal{T}_{V_1}X_1 + \mathcal{H}\mathcal{D}_{V_1}X_1, \tag{3.6}$$

$$\mathcal{D}_{X_1}V_1 = \mathcal{A}_{X_1}V_1 + \mathcal{V}\mathcal{D}_{X_1}V_1, \tag{3.7}$$

$$\mathcal{D}_{X_1}Y_1 = \mathcal{H}\mathcal{D}_{X_1}Y_1 + \mathcal{A}_{X_1}Y_1, \tag{3.8}$$

where $\widehat{\mathcal{D}}_{V_1}W_1 = \mathcal{V}\mathcal{D}_{V_1}W_1$. Moreover, if X_1 is basic, then we have $\mathcal{H}\mathcal{D}_{V_1}X_1 = \mathcal{A}_{X_1}V_1$. It appears that \mathcal{T} is acting at fibers as second fundamental form, whereas \mathcal{A} is acting at horizontal distribution and measuring obstruction to integrability of the distribution. Further details are given in the paper of O'Neill [4] in addition to this book [2].

At the end, the concept of second fundamental form of the map within Riemannian manifolds is recalled. Suppose (M, g) and (N, g_N) are Riemannian manifolds and $f: (M, g) \rightarrow (N, g_N)$ is the smooth map. Therefore, second fundamental form of f is written as

$$(\mathcal{D}f_*)(U, V) = \mathcal{D}_U^f f_*F - f_*(\mathcal{D}_U V) \tag{3.9}$$

for $U, V \in \Gamma(TM)$, with ∇^f defining the pull-back connection, and \mathcal{D} defines the Riemannian connections of the metrics g and g_N . Symmetry is widely known property of second fundamental form, and further, f is named *totally geodesic* [31] in case $(\mathcal{D}f_*)(E, F) = 0$ for any $U, V \in \Gamma(TM)$ (as in [19, p. 119]), and f is named the *harmonic map* [29] in case $trace(\mathcal{D}f_*) = 0$ (as in [19, p. 73]).

4. Anti-invariant Riemannian submersions

We first recall idea of an anti-invariant Riemannian submersion where its total manifold is the almost contact metric manifold.

Definition 4.1. ([18, 27]) *Let M be $(2n + 1)$ -dimensional almost contact metric manifold among almost contact metric constructor $(\varphi, \zeta, \eta, g)$ and N is the Riemannian manifold among Riemannian metric g_N . Considering there is Riemannian submersion $\psi: M \rightarrow N$ where vertical distribution $\ker\psi_*$ defines anti-invariant with respect to φ , which means, $\varphi\ker\psi_* \subseteq \ker\psi_*^\perp$. Therefore, Riemannian submersion π is named the anti-invariant Riemannian submersion. Similar submersions are called the anti-invariant submersions.*

Here, horizontal distribution $\ker\psi_*^\perp$ is given as.

$$\ker\psi_*^\perp = \varphi\ker\psi_* \oplus \mu, \quad (4.1)$$

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with μ refers to orthogonal complementary distribution of $\varphi\ker\psi_*$ at $\ker\psi_*^\perp$, and it is invariant with respect to φ .

It is said that the anti-invariant $\psi: M \rightarrow N$ allows vertical Reeb vector field in case Reeb vector field ξ is tangent to $\ker\psi_*$ and allows horizontal Reeb vector field in case Reeb vector field ξ is normal to $\ker\psi_*$. Clearly, μ includes Reeb vector field ξ if $\psi: M \rightarrow N$ allows horizontal Reeb vector field ξ .

Now, we begin to study anti-invariant submersions admitting vertical Reeb vector field from Lorentzian trans-Sasakian manifolds $(M, \varphi, \zeta, \eta, g)$ of type (α, β) using (nontrivial) example.

Example 4.2. Suppose M is three-dimensional Euclidean space written as

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid yz \neq 0\}.$$

We consider the Lorentzian trans-Sasakian structure $(\varphi, \zeta, \eta, g)$ at M with $\alpha = \frac{1}{2}z^2 \neq 0$ and $\beta = \frac{1}{z} \neq 0$ [23] given by the following:

$$\zeta = \frac{\partial}{\partial z}, \quad \eta = dz, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and φ is $(1, 1)$ tensor field denoted as $\varphi(E_1) = -E_2, \varphi(E_2) = -E_1, \varphi(E_3) = 0$.

An orthonormal φ -basis of this structure is written as

$$\left\{ E_1 = z \left(\frac{\partial}{\partial x} + y \frac{\partial}{\partial z} \right), \quad E_2 = z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z} \right\}.$$

Here, the map $\psi: (M, \varphi, \xi, \eta, g) \rightarrow (\mathbb{R}, g_1)$ is introduced as:

$$\psi(x, y, z) = \frac{x+y}{\sqrt{2}},$$

where g_1 is Lorentzian metric on \mathbb{R} . Therefore, Jacobian matrix of ψ is given as:

$$\begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

Since the rank of this matrix equals 1, the map π is the submersion. Using some calculation leads to

$$\ker\psi_* = \text{span} \left\{ U = - \left(\frac{E_1 + E_2}{\sqrt{2}} \right), \quad V = E_3 \right\},$$

and

$$\ker\psi_*^\perp = \text{span} \left\{ W = \frac{E_1 + E_2}{\sqrt{2}} \right\}.$$

Immediate calculations show that ψ ensures the condition **(S2)**. Thus, ψ is the Riemannian submersion. Moreover, we got $\varphi(U) = W$. Therefore, ψ is the anti-invariant submersion admitting vertical Reeb vector field.

Remark. Throughout this research, as a total manifold of an anti-invariant submersion, let us consider a Lorentzian trans-Sasakian manifold $(M, \varphi, \zeta, \eta, g)$ of type (α, β) such that both $\alpha \neq 0$ and $\beta \neq 0$.

Notion of Lagrangian submersion is considered the specific case from notion of anti-invariant submersion. We next recall the definition of the Lagrangian submersion from Lorentzian trans-Sasakian manifold onto a Riemannian manifold.

Definition 4.3. ([12]) Let ψ be the anti-invariant Riemannian submersion from the almost contact metric manifold $(M, \varphi, \xi, \eta, g)$ on the Riemannian manifold (N, g_N) . In case $\mu = \{0\}$ or $\mu = \text{span}\{\xi\}$, i.e. $\ker\psi_*^\perp = \varphi(\ker\psi_*)$ or $\ker\psi_*^\perp = \varphi(\ker\psi_*) \oplus \langle \xi \rangle$, in the same order, therefore ψ is called the Lagrangian submersion.

Let ψ is the anti-invariant submersion from the Lorentzian trans-Sasakian manifold $(M, \varphi, \zeta, \eta, g)$ on the Riemannian manifold (N, g_N) . For any $X_1 \in \ker\psi_*^\perp$, we write

$$\varphi X_1 = BX_1 + CX_1, \tag{4.2}$$

with $BX \in \Gamma(\ker\pi_*)$ and $CX \in \Gamma(\ker\pi_*^\perp)$.

Definition 4.4. ([7]) Suppose S is the revolution surface at \mathbb{R}^3 with rotation axis \mathbb{L} . For all $q \in S$, where $\gamma(q)$ represents the distance between q and \mathbb{L} . Choosing the geodesic $\sigma : U \subset \mathbb{R} \rightarrow S$ on S , Suppose $\Theta(l)$ is an angle between $\sigma(l)$ and the meridian curve through $\sigma(l)$, $l \in U$. By the famous Clairaut's theorem, we know that for all geodesics σ on S , the product $\gamma \sin\Theta$ is constant along σ , which means the independence of l .

Using geometrical analysis of Riemannian submersions, Bishop [7] described the idea of Clairaut submersion as follows.

Definition 4.5. ([7]) The Riemannian submersion $\psi: M \rightarrow N$ is known as the Clairaut submersion in case there is positive function γ at M , that is, for all geodesics σ at M , the function $(\gamma \circ \sigma) \sin\Theta$ is constant, where, for all l , $\Theta(l)$ is an angle within $\hat{\sigma}(l)$ and horizontal space $\sigma(l)$.

Bishop also provided the necessary and sufficient condition for the Riemannian submersion turns into the Clairaut submersion as follows.

Theorem 4.6. ([7]) Let $\psi: M \rightarrow N$ be the Riemannian submersion with connected fibers. Therefore ψ is the Clairaut submersion with $\gamma = \exp(\omega)$ if and only if all fibers are totally umbilical and have the mean curvature vector field $H = -\mathcal{D}\omega$, where $\mathcal{D}\omega$ is gradient of the function ω with respect to g .

5. Anti-invariant submersions admitting horizontal Reeb vector field from Lorentzian trans-Sasakian manifolds

The anti-invariant submersions are studied in this part from trans-Sasakian manifolds conceding horizontal Reeb vector field. First, the modern necessary and sufficient condition for similar submersions turns into a Clairaut submersion, and then a few distinctive outcomes for this sort of submersions are shown.

We observe from Definition 4.5, the source of the knowledge of a Clairaut submersion comes from geodesic on its total space. As a result, the necessary and sufficient condition of the curve on total space explored remains geodesic.

Now, the following results are given:

Theorem 5.1. Let $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian trans-Sasakian manifold of type (α, β) onto the Riemannian manifold allowing

horizontal Reeb vector field. In case $\sigma : U \subset \mathbb{R} \rightarrow M$ is regular curve and $V_1(\ell)$ in addition to $Z_1(\ell)$ defines vertical and horizontal components of the tangent vector field $\hat{\sigma}(\ell) = G$ of $\sigma(\ell)$, in the same order, therefore σ is geodesic if and only if through σ the following equation

$$\mathcal{V}\mathcal{D}_\delta BZ_1 + \mathcal{A}_{Z_1}\varphi V_1 + \mathcal{T}_{V_1}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1 + \alpha\eta(Z_1)V_1 + \beta\eta(Z_1)BZ_1 = 0 \quad (5.1)$$

$$\mathcal{H}\mathcal{D}_\delta CZ_1 + \mathcal{H}\mathcal{D}_\delta\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1 + \alpha[\eta(Z_1)Z_1 + v\zeta] \quad (5.2)$$

$$+ \beta[\eta(Z_1)\varphi(V_1) + \eta(Z_1)CZ_1] = 0.$$

hold, where \sqrt{s} is constant speed of σ .

Proof. In view of Eqn (2.4), we find

$$(\mathcal{D}_\delta\varphi)\hat{\sigma} = \varphi\mathcal{D}_\delta\hat{\sigma} + \alpha[g(\hat{\sigma}, \hat{\sigma})\zeta - \eta(\hat{\sigma})\hat{\sigma}] + \beta[g(\varphi\hat{\sigma}, \hat{\sigma})\zeta - \eta(\hat{\sigma})\varphi\hat{\sigma}] \quad (5.3)$$

Since $\hat{\sigma} = V_1 + Z_1$, $g(\hat{\sigma}, \hat{\sigma}) = s$, and $\eta(V_1) = 0$, we can note

$$\mathcal{D}_{V_1+Z_1}\varphi(V_1 + Z_1) = \varphi\mathcal{D}_\delta\hat{\sigma} + \alpha[v\zeta - \eta(V_1)\hat{\sigma} - \eta(Z_1)\hat{\sigma}] - \beta[\eta(Z_1)(\varphi V_1 + \varphi Z_1)]. \quad (5.4)$$

Now, from a straight forward calculation, we find

$$\begin{aligned} \mathcal{D}_{V_1}\varphi V_1 + \mathcal{D}_{V_1}\varphi Z_1 + \mathcal{D}_{Z_1}\varphi V_1 + \mathcal{D}_{Z_1}\varphi Z_1 &= \varphi\mathcal{D}_\delta\hat{\sigma} + \alpha[v\zeta - \eta(Z_1)V_1 - \eta(Z_1)Z_1] \\ &\quad - \beta[\eta(Z_1)(\varphi V_1 + \varphi Z_1)]. \end{aligned} \quad (5.5)$$

In fact $\eta(V_1) = 0$. By using Eqns (3.3), (3.4), (3.5) and (3.6), we find

$$\begin{aligned} \mathcal{H}(\mathcal{D}_\delta\varphi V_1 + \mathcal{D}_\delta CZ_1) + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})(BZ_1 + CZ_1) + \mathcal{V}\mathcal{D}_\delta BZ_1 + \mathcal{A}_{Z_1}\varphi V_1 + \mathcal{T}_{V_1}\varphi V_1 \\ = \varphi\mathcal{D}_\delta\hat{\sigma} + \alpha[v\zeta - \eta(Z_1)V_1 - \eta(Z_1)Z_1] - \beta[\eta(Z_1)BZ_1 + CZ_1 + \varphi V_1]. \end{aligned} \quad (5.6)$$

Now capturing the vertical and horizontal components from Eqn (5.6), we find the following equations:

$$\mathcal{V}\mathcal{D}_\delta BZ_1 + \mathcal{A}_{Z_1}\varphi V_1 + \mathcal{T}_{V_1}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1 = \mathcal{V}\varphi\mathcal{D}_\delta\hat{\sigma} - \alpha\eta(Z_1)V_1 - \beta\eta(Z_1)BZ_1 \quad (5.7)$$

and

$$\begin{aligned} \mathcal{H}\mathcal{D}_\delta CZ_1 + \mathcal{H}\mathcal{D}_\delta\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1 \\ = \mathcal{H}\varphi\mathcal{D}_\delta\hat{\sigma} - \alpha v\zeta - \alpha\eta(Z_1)Z_1 - \beta\eta(Z_1)\varphi V_1 - \beta\eta(Z_1)CZ_1. \end{aligned} \quad (5.8)$$

From equations (5.7) and (5.8), it is simply observed that σ is geodesic if and only if (5.1) and (5.2) hold. \square

Using Theorem (5.1) in addition to Remark (1), the following corollaries are obtained.

Corollary 5.2. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian α -Sasakian manifold of type $(\alpha, 0)$ onto the Riemannian manifold allowing horizontal Reeb vector field. In case $\sigma : U \subset \mathbb{R} \rightarrow M$ is regular curve and $V_1(\ell)$ in addition to $Z_1(\ell)$ defines vertical and horizontal components of tangent vector field $\hat{\sigma}(\ell) = G$ of $\sigma(\ell)$, in the same order, therefore σ is geodesic if and only if through σ the following equations*

$$\mathcal{V}\mathcal{D}_\delta BZ_1 + \mathcal{A}_{Z_1}\varphi V_1 + \mathcal{T}_{V_1}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1 + \alpha\eta(Z_1)V_1 = 0 \quad (5.9)$$

$$\mathcal{H}\mathcal{D}_{\hat{\sigma}}CZ_1 + \mathcal{H}\mathcal{D}_{\hat{\sigma}}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1 + \alpha[\eta(Z_1)Z_1 + v\zeta] = 0. \quad (5.10)$$

maintain, where \sqrt{s} is constant speed of σ .

Corollary 5.3. Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian β -Kenmotsu manifold of type $(0, \beta)$ onto the Riemannian manifold admitting horizontal Reeb vector field. In case $\sigma: U \subset \mathbb{R} \rightarrow M$ is the regular curve and $V_1(l)$ in addition to $Z_1(l)$ defines vertical and horizontal components of tangent vector field $\hat{\sigma}(l) = G$ of $\sigma(l)$, in the same order, therefore σ is geodesic if and only if through σ the following equation

$$v\mathcal{D}_{\hat{\sigma}}BZ_1 + \mathcal{A}_{Z_1}\varphi V_1 + \mathcal{T}_{V_1}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1 + \beta\eta(Z_1)BZ_1 = 0 \quad (5.11)$$

$$\mathcal{H}\mathcal{D}_{\hat{\sigma}}CZ_1 + \mathcal{H}\mathcal{D}_{\hat{\sigma}}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1 + \beta[\eta(Z_1)\varphi(V_1) + \eta(Z_1)CZ_1] = 0. \quad (5.12)$$

hold, where \sqrt{s} is constant speed of σ .

Corollary 5.4. Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is an anti-invariant Riemannian submersion from Lorentzian cosymplectic manifold of type $(0, 0)$ onto the Riemannian manifold allowing horizontal Reeb vector field. If $\sigma: U \subset \mathbb{R} \rightarrow M$ is the regular curve and $V_1(l)$ in addition to $Z_1(l)$ defines vertical and horizontal components of the tangent vector field $\hat{\sigma}(l) = G$ of $\sigma(l)$, in the same order, therefore σ is geodesic if and only if through σ the following equation

$$v\mathcal{D}_{\hat{\sigma}}BZ_1 + \mathcal{A}_{Z_1}\varphi V_1 + \mathcal{T}_{V_1}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})CZ_1 = 0 \quad (5.13)$$

$$\mathcal{H}\mathcal{D}_{\hat{\sigma}}CZ_1 + \mathcal{H}\mathcal{D}_{\hat{\sigma}}\varphi V_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1 = 0. \quad (5.14)$$

hold, where \sqrt{s} is constant speed of σ .

Theorem 5.5. Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian trans-Sasakian manifold of type (α, β) onto the Riemannian manifold allowing horizontal Reeb vector field. Therefore ψ is Clairaut submersion with $\gamma = \exp(\omega)$ if and only if through σ

$$[g(\mathcal{D}\omega, Z_1) - \beta\eta(Z_1)]|V_1|^2 = g(\alpha\eta(Z_1)Z_1 + \mathcal{H}\mathcal{D}_{\hat{\sigma}}CZ_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \varphi V_1) \quad (5.15)$$

holds, where $V_1(l)$ and $Z_1(l)$ are vertical and horizontal components of the tangent vector field $\hat{\sigma}(l)$ of the geodesic $\sigma(l)$ at M , in the same order.

Proof. Consider $\sigma(l)$ as the geodesic having the speed \sqrt{s} at M , therefore,

$$s = |\hat{\sigma}(l)|^2. \quad (5.16)$$

Now, from Eqn (5.16), we achieve that

$$g(V_1(l), V_1(l)) = v\text{Sin}^2\Theta(l) \quad \text{and} \quad g(Z_1(l), Z_1(l)) = v\text{Cos}^2\Theta(l), \quad (5.17)$$

where $\Theta(l)$ is the angle within $\hat{\sigma}(l)$ and horizontal space at $\sigma(l)$. Now, by the derivative of first part of Eqn (5.17), we find

$$\frac{d}{dl}g(V_1(l), V_1(l)) = 2g(\mathcal{D}_{\hat{\sigma}(l)}V_1(l), V_1(l)) = 2v\text{Sin}\Theta\text{Cos}\Theta\frac{d\Theta}{dl}(l), \quad (5.18)$$

Using the Lorentzian trans-Sasakian structure, we find

$$g(\varphi\mathcal{D}_{\hat{\sigma}(l)}V_1(l), \varphi V_1(l)) = v\text{Sin}\Theta\text{Cos}\Theta\frac{d\Theta}{dl}(l), \quad (5.19)$$

Once again, from Eqn (2.4), we have

$$\varphi\mathcal{D}_\delta V_1 = \mathcal{D}_\delta\varphi V_1 - \alpha g(\hat{\alpha}, V_1)\zeta - \beta g(\varphi\hat{\sigma}, V_1)\zeta. \quad (5.20)$$

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Hence,

$$g(\varphi\mathcal{D}_\delta V_1, \varphi V_1) = g(\mathcal{H}\varphi\mathcal{D}_\delta V_1, \varphi V_1), \quad (5.21)$$

since $\eta(V) = 0$, $g(\varphi V_1, \zeta) = 0$ and using the fact that φV_1 is horizontal.

Thus, from Eqn (5.19), we obtain

$$g(\mathcal{H}\varphi\mathcal{D}_\delta V_1, \varphi V_1) = v\text{Sin}\Theta\text{Cos}\Theta \frac{d\Theta}{dl}. \quad (5.22)$$

From Eqn (5.2), we find along σ ,

$$-g(\mathcal{H}\mathcal{D}_\delta CZ_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1 + \alpha\eta(Z_1)Z_1 + \beta\eta(Z_1)\varphi(V_1), \varphi V_1) = v\text{Sin}\Theta\text{Cos}\Theta \frac{d\Theta}{dl}, \quad (5.23)$$

since $g(\varphi V_1, \zeta) = 0$.

On contrary, ψ is Clairaut submersion with $\gamma = \exp(\omega)$ if and only if

$$\frac{d}{dl}[\exp(\omega)\text{Sin}\Theta] = 0 \quad \leftrightarrow \quad \exp(\omega) \left[\frac{d\omega}{dl} \text{Sin}\Theta + \text{Cos}\Theta \frac{d\Theta}{dl} \right] = 0. \quad (5.24)$$

Now, taking the product of Eqn (5.24) with nonzero factor $v\text{Sin}\Theta$, we find

$$\frac{d\omega}{dl} v\text{Sin}^2\Theta + v\text{Sin}\Theta\text{Cos}\Theta \frac{d\Theta}{dl} = 0. \quad (5.25)$$

Using equations (5.23) and (5.24), we obtain

$$\frac{d\omega}{dl} [\sigma(l)] |V_1|^2 = g(\alpha\eta(Z_1)Z_1 + \mathcal{H}\mathcal{D}_\delta CZ_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \varphi V_1) + \beta\eta(Z_1) |V_1|^2. \quad (5.26)$$

In fact $\frac{d\omega}{dl} [\sigma(l)] = \hat{\sigma}[\omega] = g(\mathcal{D}\omega, \hat{\sigma}) = g(\mathcal{D}\omega, Z_1)$, the expression (5.29) follows from (5.26). \square

Now, the following corollaries are given:

Corollary 5.6. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian α -Sasakian manifold of type $(\alpha, 0)$ onto the Riemannian manifold allowing horizontal Reeb vector field. Therefore ψ is Clairaut submersion with $\gamma = \exp(\omega)$ if and only if through σ*

$$[g(\mathcal{D}\omega, Z_1)] |V_1|^2 = g(\alpha\eta(Z_1)Z_1 + \mathcal{H}\mathcal{D}_\delta CZ_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \varphi V_1) \quad (5.27)$$

holds, where $V_1(l)$ and $Z_1(l)$ are vertical and horizontal components of the tangent vector field $\hat{\sigma}(l)$ of the geodesic $\sigma(l)$ at M , in the same order.

Corollary 5.7. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian β -Kenmotsu manifold of type $(0, \beta)$ onto the Riemannian manifold allowing horizontal Reeb vector field. Therefore ψ is Clairaut submersion with $\gamma = \exp(\omega)$ if and only if through σ*

$$[g(\mathcal{D}\omega, Z_1) - \beta\eta(Z_1)] |V_1|^2 = g(\mathcal{H}\mathcal{D}_\delta CZ_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \varphi V_1) \quad (5.28)$$

holds, where $V_1(l)$ and $Z_1(l)$ are vertical and horizontal components of tangent vector field $\hat{\sigma}(l)$ of the geodesic $\sigma(l)$ at M , in the same order.

Corollary 5.8. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the anti-invariant Riemannian submersion from Lorentzian cosymplectic manifold of type (α, β) onto the Riemannian manifold admitting horizontal Reeb vector field. Therefore ψ is Clairaut submersion with $\gamma = \exp(\omega)$ if and only if along σ*

$$g(\mathcal{D}\omega, Z_1)|V_1|^2 = g(\mathcal{H}\mathcal{D}_{\hat{\sigma}}CZ_1 + (\mathcal{T}_{V_1} + \mathcal{A}_{Z_1})BZ_1, \varphi V_1) \tag{5.29}$$

holds, where $V_1(l)$ and $Z_1(l)$ are vertical and horizontal components of the tangent vector field $\hat{\sigma}(l)$ of the geodesic $\sigma(l)$ at M , in the same order.

Now, from Eqn (5.29), we also obtain the following conclusion.

Corollary 5.9. *Suppose ψ is the Clairaut anti-invariant submersion from Lorentzian trans-Sasakian manifold $(M, \varphi, \zeta, \eta, g)$ of type (α, β) on the Riemannian manifold (N, g_N) . Therefore,*

$$g(\mathcal{D}\omega, \zeta) = \beta. \tag{5.30}$$

Proof. Since ζ is a horizontal Reeb vector field. Setting $Z_1 = \zeta$ and using the fact $\frac{d\omega}{dl}[\sigma(l)] = \hat{\sigma}[\omega] = g(\mathcal{D}\omega, \hat{\sigma}) = g(\mathcal{D}\omega, Z_1)$, the expression (5.26) gives (5.30). \square

Corollary 5.10. *Suppose ψ is the Clairaut anti-invariant submersion from Lorentzian α -Sasakian (or Lorentzian Sasakian) manifold $(M, \varphi, \zeta, \eta, g)$ of type $(\alpha, 0)$ onto the Riemannian manifold (N, g_N) . Therefore,*

$$g(\mathcal{D}\omega, \zeta) = 0. \tag{5.31}$$

Proof. Since for Lorentzian α -Sasakian (or Lorentzian Sasakian) $\beta = 0$, and using similar fact as we have used in proof of Corollary 5.9 together, we find the desired result. \square

Theorem 5.11. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is a Clairaut anti-invariant submersion from Lorentzian trans-Sasakian manifold of type (α, β) onto a Riemannian manifold admitting horizontal Reeb vector field with $\gamma = \exp(\omega)$. Then we have*

$$\mathcal{A}_{\varphi V_1}\varphi G_1 = G_1(\omega)V_1 \tag{5.32}$$

for $G_1 \in \Gamma(\mu)$ and $V_1 \in (\ker\psi^*)$ such that φV_1 is basic vector.

Proof. Suppose ψ is the Clairaut anti-invariant submersion allowing horizontal Reeb vector field from a Lorentzian trans-Sasakian manifold onto a Riemannian manifold with $\gamma = \exp(\omega)$. Now, by consequences of Theorem (4.6), we find

$$\mathcal{T}_{U_1}G_1 = -g(U_1, G_1)\mathcal{D}\omega \tag{5.33}$$

for $U_1, G_1 \in (\ker\psi^*)$. If we spread Eqn (5.33) with $\varphi V_1, V_1 \in (\ker\psi^*)$ such that φV_1 is basic and using Eqn (3.3), we find

$$g(\mathcal{D}_{U_1}G_1, \varphi V_1) = -g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \tag{5.34}$$

$$g(\mathcal{D}_{U_1}\varphi V_1, G_1) = g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \tag{5.35}$$

In fact $g(G_1, \varphi V_1) = 0$. Through Eqn (2.4), we infer

$$g(\varphi\mathcal{D}_{U_1}V_1, G_1) = -g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \tag{5.36}$$

Adopting the Lorentzian trans-Sasakian structure, we notice

$$-g(\mathcal{D}_{U_1} V_1, \varphi G_1) = g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \quad (5.37)$$

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Once again, adopting (3.3), we turn up

$$-g(\mathcal{T}_{U_1} V_1, \varphi G_1) = g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \quad (5.38)$$

Henceforth, through Eqn (5.33), we attain

$$g(U_1, V_1)g(\mathcal{D}\omega, \varphi G_1) = g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \quad (5.39)$$

Putting $U_1 = V_1$ and shifting U_1 with by G_1 in Eqn (5.39), we acquire

$$|G_1|^2 g(\mathcal{D}\omega, \varphi G_1) = g(U_1, G_1)g(\mathcal{D}\omega, \varphi V_1). \quad (5.40)$$

Adopting Eqn (5.39) with setting $V_1 = U_1$, we have

$$g(\mathcal{D}\omega, \varphi G_1) = \frac{g^2(U_1, G_1)}{|U_1|^2 |G_1|^2} g(\mathcal{D}\omega, \varphi V_1). \quad (5.41)$$

On the contrary, involving Eqn (2.4), we turn up

$$g(\mathcal{D}_{G_1} \varphi V_1, \varphi W_1) = g(\varphi \mathcal{D}_{G_1}, \varphi W_1). \quad (5.42)$$

for $W_1 \in \Gamma(\mu)$ and $W_1 \neq \zeta$. Using Eqn (2.5), we get

$$g(\mathcal{D}_{G_1} \varphi V_1, \varphi W_1) = g(\mathcal{D}_{G_1}, W_1). \quad (5.43)$$

Adopting equations (3.3) and (5.33), we get

$$g(\mathcal{D}_{G_1} \varphi V_1, \varphi W_1) = g(V_1, \varphi G_1)g(\mathcal{D}\omega, W_1). \quad (5.44)$$

After all φV_1 is basic vector and using the case that $\mathcal{H}\mathcal{D}_{G_1} \varphi V_1 = \mathcal{A}_{\varphi V_1} G_1$, we turn up

$$g(\mathcal{D}_{G_1} \varphi V_1, \varphi W_1) = g(\mathcal{A}_{\varphi G_1} V_1, \varphi W_1). \quad (5.45)$$

Involving again, Eqns (5.44), (5.45) and the skew-symmetric nature of A , we turn up

$$g(\mathcal{D}\omega, W_1)g(V_1, \varphi G_1) = g(\mathcal{A}_{\varphi G_1} V_1, \varphi W_1). \quad (5.46)$$

By reason of $\mathcal{A}_{\varphi V_1} \varphi W_1$, G_1 and V_1 are vertical and ω is horizontal, we turn up expression (5.32).

Particularly if $\mathcal{D}\omega \in \varphi(\ker \psi_*)$, then from (5.41) in proof of Theorem 5.11 and the equality case of Schwarz inequality, we have have that \square

Corollary 5.12. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the Clairaut Lagrangian submersion allowing horizontal Reeb vector field from Lorentzian trans-Sasakian manifold of type (α, β) onto a Riemannian manifold with $\gamma = \exp(\omega)$. If $\mathcal{D}\omega \in \varphi(\ker \psi_*)$, then either ω is constant on $\varphi(\ker \psi_*)$ or fiber of ψ is one-dimensional.*

6. Clairaut Lagrangian submersions

This section deals with some results of Clairaut Lagrangian submersions conceding with horizontal Reeb vector field. Moreover, when the function ω is constant, $\mathcal{D}\omega = 0$. Thus by Theorem 4.6 and Corollary 5.12, we have the following results.

Corollary 6.1. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is a Clairaut anti-invariant submersion allowing horizontal Reeb vector field from Lorentzian trans-Sasakian manifold of type (α, β) on the Riemannian manifold with $\gamma = \exp(\omega)$ and $\dim(\ker\psi_*) > 1$, then fibers of ψ are totally geodesic if and only if*

$$\mathcal{A}_{\varphi V_1} \varphi Z_1 = 0$$

for $V_1 \in (\ker\psi_*)$, φV_1 is basic and $Z_1 \in \mu$.

Moreover, in case the submersion ψ at Theorem (5.11) is Lagrangian submersion, therefore $\mathcal{A}_{\varphi V_1} \varphi Z_1$ is always vanish, because $\mu = \{0\}$ or $\mu = \text{span}\{\zeta\}$. Also from Corollaries 5.9 and 5.10, we have $D\omega \in \varphi(\ker\psi_*)$. Hence, the following consequences of Theorem (5.11) and Corollary 5.12 are given.

Theorem 6.2. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the Clairaut Lagrangian submersion allowing horizontal Reeb vector field from Lorentzian trans-Sasakian manifold of type (α, β) onto a Riemannian manifold with $\gamma = \exp(\omega)$. Therefore, fibers of ψ can be one-dimensional or totally geodesic.*

Corollary 6.3. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the Clairaut Lagrangian submersion admitting horizontal Reeb vector field from Lorentzian α -Sasakian manifold of type $(\alpha, 0)$ onto a Riemannian manifold with $\gamma = \exp(\omega)$. Therefore fibers of ψ can be one-dimensional or totally geodesic.*

Corollary 6.4. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the Clairaut Lagrangian submersion allowing horizontal Reeb vector field from Lorentzian β -Kenmotsu manifold of type $(0, \beta)$ onto a Riemannian manifold with $\gamma = \exp(\omega)$. Therefore fibers of ψ can be one-dimensional or totally geodesic.*

Corollary 6.5. *Suppose $\psi: (M, \varphi, \zeta, \eta, g) \rightarrow (N, g_N)$ is the Clairaut Lagrangian submersion allowing horizontal Reeb vector field from Lorentzian cosymplectic manifold of type (α, β) onto a Riemannian manifold with $\gamma = \exp(\omega)$. Therefore either fibers of ψ can be one-dimensional or totally geodesic.*

7. Applications

The following result is Theorem 2 stated by Gauchman in [28].

Theorem 7.1. *Suppose $\psi: (M, g) \rightarrow (N, g_N)$ is the Clairaut submersion with γ , where M is complete, connected and simply connected, and N is simply connected. Assume that any vertical leaf of ψ has no nontrivial Killing vector field. Suppose p is the point of M . Therefore M is isometric to the warped product $N \times_f B$, where B is the vertical leaf through p and $f: N \rightarrow \mathbb{R}$ is determined using this equation $\gamma = f \circ \psi$.*

In [23] De and Srakar prove that trans-Sasakian structures are complete and connected. Indeed, Riemannian manifold also preserved the characteristic of simple connectedness. Therefore, the following results are obtained.

Theorem 7.2. *$\psi: (M, \varphi, \zeta, \eta, g, \alpha, \beta) \rightarrow (N, g_N)$ is a Clairaut Lagrangian submersion with γ , where $(M, \varphi, \zeta, \eta, g)$ is complete, connected, and simply connected Lorentzian trans-Sasakian manifold, and Riemannian manifold (N, g_N) is simply connected. Assume that any vertical leaf of ψ has no nontrivial Killing vector field. Let p be a point of $(M, \varphi, \zeta, \eta, g)$. Then Lorentzian trans-Sasakian manifold of (α, β) type is isometric to a warped product $N \times_f B$, where B is the vertical leaf through p and $f: N \rightarrow \mathbb{R}$ is defined by the equation $\gamma = f \circ \psi$.*

Remark. *For particular values of α and β easily we can turn up the similar results like Theorem (7.2) for α -Lorentzian Sasakian manifold (Lorentzian Sasakian manifold),*

β -Lorentzian Kenmotsu manifold (Lorentzian Kenmotsu manifold), and Lorentzian cosymplectic manifold.

Clairaut anti-invariant submersions

Now, we describe some examples of Clairaut submersion from Lorentzian trans-Sasakian manifolds $(M, \varphi, \xi, \eta, g)$ of type (α, β) .

Example 7.3. Suppose M is three-dimensional Euclidean space written as

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid yz \neq 0\}.$$

We consider the Lorentzian trans-Sasakian structure $(\varphi, \xi, \eta, g, \alpha, \beta)$ at M with $\alpha = 0$ and $\beta = 1$ [23] given by the following:

$$\zeta = \frac{\partial}{\partial z}, \quad \eta = dz, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and φ is the $(1, 1)$ tensor field determined as

$\varphi(E_1) = -E_2, \varphi(E_2) = -E_1, \varphi(E_3) = 0$. An orthonormal φ -basis of this constructor is written as

$$\left\{ E_1 = z \frac{\partial}{\partial x}, \quad E_2 = z \frac{\partial}{\partial y}, \quad E_3 = z \frac{\partial}{\partial z} \right\}.$$

Here, the map $\psi : (M, \varphi, \xi, \eta, g, \alpha, \beta) \rightarrow (\mathbb{R}, g_1)$ is written as:

$$\psi(x, y, z) = \left(\frac{x+y}{\sqrt{2}}, z \right),$$

where g_1 is the usual metric at \mathbb{R} . Now, by a straightforward computation, we turn up

$$\ker \psi_* = \text{span} \left\{ U = - \left(\frac{E_1 + E_2}{\sqrt{2}} \right) \right\},$$

and

$$\ker \psi_*^\perp = \text{span} \left\{ V = \frac{E_1 + E_2}{\sqrt{2}}, \quad W = E_3 \right\}.$$

Easily, we observe that ψ is the Riemannian submersion. Moreover, we have $\varphi(U) = V$. Therefore, ψ is the anti-invariant submersion allowing horizontal Reeb vector field. Particularly, ψ is Lagrangian submersion. Furthermore, after all the fibers of ψ are one-dimensional, then they are simply totally umbilical. At this point, it is proved that fibers are not considered totally geodesic, and it is found that the function of \mathbb{R}^3 obeying $T_{U_1}U_1 = -D\omega$. Therefore, after some sort of calculation, we turn up

$$\mathcal{D}_{U_1}U_1 = \frac{1}{2} (\mathcal{D}_{E_1}E_1 - \mathcal{D}_{E_1}E_2 - \mathcal{D}_{E_2}E_1 - \mathcal{D}_{E_2}E_2). \quad (7.1)$$

Adopting the Lorentzian trans-Sasakian structure results in

$$\mathcal{D}_{E_1}E_1 = \mathcal{D}_{E_2}E_2 = -E_3 \quad \text{and} \quad \mathcal{D}_{E_1}E_2 = -\mathcal{D}_{E_2}E_1 = 0$$

$$\mathcal{D}_{U_1}U_1 = -z \frac{\partial}{\partial z}$$

Using (3.5), we turn up

$$\mathcal{T}_{U_1}U_1 = -z \frac{\partial}{\partial z}.$$

For any function ω of $(\mathbb{R}^3, \varphi, \zeta, \eta, g)$, the gradient of ω with respect to the metric g is

$$\mathcal{D}\omega = \sum_{i,j}^3 \frac{\partial \omega}{\partial x_i} \frac{\partial}{\partial x_j} = \left[\frac{\partial \omega}{\partial x} \frac{\partial}{\partial x} + \frac{\partial \omega}{\partial y} \frac{\partial}{\partial y} - \frac{\partial \omega}{\partial z} \frac{\partial}{\partial z} \right].$$

Here, at this point, it is clear to observe that $\omega = -\frac{z^2}{2}$ for the function of z and $\mathcal{T}_{U_1}U_1 = -D\omega = -\zeta$. Also for any $U_2 \in (\ker\psi)_*$, we have

$$\mathcal{T}_{U_2}U_2 = -|U_2|^2 \mathcal{D}\omega.$$

Henceforth, using Theorem (5.26), the submersion ψ is Clairaut submersion.

Example 7.4. Suppose M is three-dimensional Euclidean space written as

$$M = \{(x, y, z) \in \mathbb{R}^3 \mid yz \neq 0\}$$

We consider the Lorentzian trans-Sasakian structure $(\varphi, \xi, \eta, g, \alpha, \beta)$ at M with $\alpha = -1$ and $\beta = 0$ given by the following:

$$\zeta = \frac{\partial}{\partial z}, \quad \eta = dz, \quad g = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and φ is (1, 1) tensor field determined as

$\varphi(E_1) = -E_1, \varphi(E_2) = -E_2, \varphi(E_3) = 0$. An orthonormal φ -basis is written as

$$\left\{ E_1 = e^{x+z} \frac{\partial}{\partial x}, \quad E_2 = e^{y+z} \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z} \right\}.$$

Moreover, we have

$$\mathcal{D}_{E_i}E_3 = -E_i, \forall i = 1, 2, \quad \mathcal{D}_{E_i}E_i = -2E_3, \forall i = 1, 2, \quad \mathcal{D}_{E_i}E_j = 0, i \neq j \text{ and } i = j = 3.$$

Here, the map $\psi : (M, \varphi, \zeta, \eta, g, \alpha, \beta) \rightarrow (\mathbb{R}, g_1)$ is defined by the following:

$$\psi(x, y, z) = \left(\frac{x+y}{\sqrt{2}}, z \right),$$

where g_1 is usual metric at \mathbb{R} . Now, by a straightforward computation, we turn up

$$\ker\psi_* = \text{span} \left\{ U = -\left(\frac{E_1 + E_2}{\sqrt{2}} \right) \right\},$$

and

$$\ker\psi_*^\perp = \text{span} \left\{ V = \frac{E_1 + E_2}{\sqrt{2}}, \quad W = E_3 \right\}.$$

Easily, we observe that ψ is the Riemannian submersion. Moreover, we have $\varphi(U) = V$. Therefore, ψ is the anti-invariant submersion admitting horizontal Reeb vector field.

Particularly, ψ is Lagrangian submersion. Furthermore, after all the fibers of ψ are one-dimensional, then they are simply totally umbilical. At this point, it is proved that fibers are not totally geodesic, and it is found that the function of \mathbb{R}^3 obeying $\mathcal{T}_{U_1}U_1 = -D\omega$. Therefore, after some sort of calculation, we turn up

$$\mathcal{D}_{U_1}U_1 = \frac{1}{2}(\mathcal{D}_{E_1}E_1 - \mathcal{D}_{E_1}E_2 - \mathcal{D}_{E_2}E_1 - \mathcal{D}_{E_2}E_2). \quad (7.2)$$

Adopting the Lorentzian trans-Sasakian structure, we observe that

$$\mathcal{D}_{U_1}U_1 = 2\frac{\partial}{\partial z}$$

Using (3.5), we turn up

$$\mathcal{T}_{U_1}U_1 = 2\frac{\partial}{\partial z}.$$

For all functions ω at $(\mathbb{R}^3, \varphi, \xi, \eta, g)$, the gradient of ω with respect to the metric g is

$$\mathcal{D}\omega = \sum_{ij}^3 \frac{\partial\omega}{\partial x_i} \frac{\partial}{\partial x_j} = \left[\frac{\partial\omega}{\partial x} \frac{\partial}{\partial x} + \frac{\partial\omega}{\partial y} \frac{\partial}{\partial y} - \frac{\partial\omega}{\partial z} \frac{\partial}{\partial z} \right].$$

Now, at this point, it is clear to observe that $\omega = -2z$ for the function of z and $\mathcal{T}_{U_1}U_1 = -D\omega = -2\xi$. Also for any $U_2 \in (\ker\psi^*)$, we have

$$\mathcal{T}_{U_2}U_2 = -|U_2|^2 D\omega.$$

Henceforth, by Theorem (5.26), the submersion ψ is Clairaut submersion.

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