Expected idiosyncratic entropy

Mohammadreza Tavakoli Baghdadabad
Business School, Western Sydney University, Parramatta, Australia

Abstract

Purpose – We propose a risk factor for idiosyncratic entropy and explore the relationship between this factor and expected stock returns.

Design/methodology/approach – We estimate a cross-sectional model of expected entropy that uses several common risk factors to predict idiosyncratic entropy.

Findings – We find a negative relationship between expected idiosyncratic entropy and returns. Specifically, the Carhart alpha of a low expected entropy portfolio exceeds the alpha of a high expected entropy portfolio by −2.37% per month. We also find a negative and significant price of expected idiosyncratic entropy risk using the Fama-MacBeth cross-sectional regressions. Interestingly, expected entropy helps us explain the idiosyncratic volatility puzzle that stocks with high idiosyncratic volatility earn low expected returns.

Originality/value – We propose a risk factor of idiosyncratic entropy and explore the relationship between this factor and expected stock returns. Interestingly, expected entropy helps us explain the idiosyncratic volatility puzzle that stocks with high idiosyncratic volatility earn low expected returns.

Keywords Entropy, Idiosyncratic entropy, Cross section, Expected return

Paper type Research paper

Received 8 March 2023
Revised 8 December 2023
Accepted 4 February 2024

1. Introduction

The field of risk management has extensively studied how the risk related to the returns of assets influences investor decisions. Research by authors like Rubinstein (1973) and Kraus and Litzenberger (1976) introduced models to predict asset returns by factoring in skewness – a measure of asymmetry in returns distribution. These models effectively link the performance of individual assets to broader market risks. They demonstrate that both the shape and the shared tendencies of return distributions play a vital role in setting asset prices (for instance, Amaya, Christoffersen, Jacobs, & Vasquez, 2015). Notably, Harvey and Siddique (2000) discovered that the interplay between individual asset returns and overall market trends (co-skewness) has a measurable financial impact within the framework of market-wide decision models.

Considering the importance of shape and shared characteristics in asset pricing, its been observed that to benefit from these characteristics, some investors opt not to diversify their investments fully. This makes the unique characteristics of each asset, such as idiosyncratic volatility (IV) and idiosyncratic skewness (IS), important (e.g. Ang, Hodrick, Xing, & Zhang, 2006; Boyer, Mitton, & Vorkink, 2010). Ang, Hodrick, Xing, and Zhang (2009) discovered that IV provides key insights into the likely future performance of investments. Similarly, studies by Brunnermeier, Gollier, and Parker (2007) and Barberis and Huang (2008) show that the asymmetry in returns of individual assets (skewness) can sway investor choices. Furthermore, Boyer *et al.* (2010) found that investments with a high expected IS often yield lower future returns.

JEL Classification — G11, G12

China Accounting and Finance Review
Emerald Publishing Limited
e-ISSN: 2307-3055
p-ISSN: 1029-807X
DOI 10.1108/CAFR-03-2023-0021

Generally, there are two strands in the literature: (1) studies on the role of risk-related statistical measures in influencing probability distributions, and (2) studies on the specific properties of these statistical measures. The second category reveals that many analyses overlook the erratic patterns in the residual, or leftover, returns when predicting stock performance. These erratic patterns in returns can appear in two ways: (1) as shifts in common statistical measures like standard deviation, skewness, and kurtosis, and (2) as various kinds of irregularities or disorders. It's important to differentiate between these statistical measures and 'entropy,' a concept leading to different insights about forecasting residual returns. Benedetto, Giunta, and Mastroeni (2016) argue that deviations from the average return don't always indicate unpredictability. The presence and number of irregularities significantly impact the prediction process, leading to unpredictability. They found that predictive methods are effective even with high deviations from the mean, provided the series has few irregularities. This suggests that series with fewer irregularities offer better prospects for predicting asset returns, an angle not fully explored in previous studies. Backus, Chernov, and Martin (2011) discovered that using entropy in asset pricing models improves the predictability of returns. They noted that deviations from typical distribution patterns, like sudden jumps, make entropy valuable in these models. Backus, Boyarchenko, and Chernov (2018) introduced 'coentropy,' which combines the effects of pricing mechanisms and cash flows. This concept emphasizes the impact of unusual market movements on risk premiums. Billio, Casarin, Costola, and Pasqualini (2016) explored systemic risk in Europe using different entropy metrics, highlighting their predictive power for financial crises. Ghosh, Julliard, and Taylor (2017) found that entropy can reveal time series details of both the Stochastic Discount Factor (SDF) and its unobservable components, overlooked by other statistical measures. They showed that entropy adds extra information to the SDFs for accurate asset return pricing. It focuses on the second-moment deviations (variance) and other measures (skewness and kurtosis), and they found that skewness and kurtosis of stock returns drive a significant portion of the entropy in their pricing model. However, combining all statistical measures in a unified pricing model captures all features of stock return variability. Calomiris and Mamaysky (2019) demonstrated that a concise summary of news, including the uniqueness (entropy) of word flow, forecasts future country-level returns, volatilities, and drawdowns. Similarly, Glasserman and Mamaysky (2019) found that the uniqueness (entropy) of word combinations can predict market outcomes, particularly when combined with sentiment analysis.

Entropy, a concept originating from both classical mechanics and information theory, plays a crucial role in understanding systems. In classical mechanics, it measures the level of uncertainty and disarray in moving systems, representing how random these systems are (Jaynes, 1965). In the realm of information theory, entropy is used to gauge the amount of information in time-based data sequences (Shannon, 1948). Shannon (1948) specifically noted that this measure indicates how uncertain a data source is when choosing what message to send, reflecting the unpredictability of the data sequence's future behavior. In practical terms, entropy is inversely related to predictability: higher entropy means a data sequence is less predictable, while lower entropy suggests greater predictability. In our study, we apply entropy metrics to examine inconsistencies in data sequences and determine their predictability.

In this paper, we introduce a new risk factor called idiosyncratic entropy (IE) and investigate how it affects stock pricing. Although theoretical frameworks like those by Gulko (1999, 2002) suggest exploring the impact of entropy on pricing, actual studies linking IE with stock returns are more intricate. A key challenge is that entropy, unlike more stable measures such as variance, fluctuates over time, making it challenging to evaluate (Maasoumi & Racine, 2002). This necessitates using additional risk factors beyond past entropy data to accurately predict future entropy levels. Echoing the approach of Chen, Hong, and Stein (2001), we incorporate various firm-specific risk factors to forecast IE. While past entropy data is a good indicator of future levels, other firm-specific factors are crucial for predicting IE, including IV and IS – both of which are gaining attention in asset pricing. Our model also considers other vital risk factors, such as idiosyncratic kurtosis (IK, as studied by Conrad, Dittmar, & Ghysels, 2013), market beta, company size, book-to-market ratio (as per Fama & French, 1993), momentum (Carhart, 1997), co-skewness (Harvey & Siddique, 2000), liquidity (Pastor & Stambaugh, 2003), and the MAX and MIN factors introduced by Bali, Cakici, and Whitelaw (2011). This prediction model allows us to observe how expected entropy varies across different stocks and over time. Moreover, the patterns in both expected and actual entropy over time seem to mimic the episodic nature often seen in IV.

In our study using the expected entropy model, we discover a consistent and strong negative correlation between average stock returns and predicted IE. By organizing stocks into groups based on their expected IE levels, we note that stocks with lower expected IE have higher average returns than those with higher expected IE, showing a difference of −2.42% per month. This difference becomes more noticeable after accounting for risk factors. Additionally, we observe that the return (measured by Carhart alpha, 1997) of the low expected IE group surpasses that of the high IE group by −2.47% monthly. Employing the Fama and MacBeth (1973) methodology, our results validate the impact of IE on stock pricing, revealing that expected IE significantly clarifies the differences in stock returns. This influence of IE is not only statistically significant but also holds up under various validation tests. In our comparative analysis, similar to findings in Ang *et al.* (2006) and Boyer *et al.* (2010), we identify a negative relationship between IV and returns, and between IS and returns. The negative correlations observed with IV and IS lend credibility to our findings regarding IE, a relationship not previously explored in the literature. Further analysis to separate the effects of IE and IV indicates that each type of risk individually provides considerable insight into predicting stock returns.

In addition, we investigate whether changes over time in predicted entropy are linked to similar variations in the anticipated rewards for taking on entropy-related risks. Our analysis shows that the times with the most substantial negative impacts on the expected entropy reward correspond to times when the expected entropy is both high and widely varied. These are also the times when our entropy prediction model is most accurate, as shown by the high R^2 values. A possible explanation for these patterns is that entropy's impact on pricing is most pronounced during these periods, offering investors who favor entropy a better chance to justify their choice of stocks with higher growth potential.

Considering the impact of IE on pricing, we explore whether entropy can shed light on the IV puzzle, where stocks with higher IV tend to have lower expected returns. This inverse relationship between IV and future returns, documented by Ang *et al.* (2006) in the U.S. and Ang *et al.* (2009) internationally, is intriguing as it contradicts current theories [1]. On one hand, theories suggest that IV should not influence pricing if an investor diversifies their portfolio completely. On the other hand, if an investor doesn't diversify fully, IV should theoretically lead to higher expected returns, as per Boyer *et al.* (2010). Yet, our discovery that IV is a key indicator of IE suggests a new perspective: investors might be drawn to high-IV stocks not for their volatility, but rather for their 'lottery-like' return characteristics, which may lead them to accept lower average returns. Our results show that different proxies for lottery-like payoffs generate similar results, providing further support for the explanation we offer.

To understand how entropy preference influences the IV puzzle, we conducted tests using our model that anticipates entropy. We look at how expected returns and IV are related when we factor in our predictions of entropy. Using a technique similar to the one used by Ang *et al.* (2009), we create portfolio groups that vary widely in IV but not in predicted IE. After adjusting for predicted IE, we see a weaker connection between IV and stock returns. In particular, the difference in average returns between portfolios with high IV and those with low IV becomes smaller and not statistically significant. Additionally, the difference in the

Carhart alphas for these portfolios decreases from -2.47% to -2.14% monthly. Considering entropy preferences is crucial for understanding why stocks with high IV often have lower average returns. We also try to separate the impacts of predicted entropy and IV. The negative relationship between predicted IE and average returns seems to be more solidly backed by both empirical data and theoretical reasoning.

The structure of this paper is outlined as follows: Section 2 delves into the reasoning behind our empirical analyses, emphasizing the theoretical connections between a preference for entropy and the expected returns of investments. In Section 3, we introduce a model for predicting entropy and discuss the results of its estimations. Section 4 presents data. Section 5 analyses the cross-sectional distribution of firm-level risk factors. Section 6 explores the effects of IE on stock pricing. Section 7 differentiates between the impacts of IE and IV, and investigates whether a preference for entropy can explain why there is often a negative relationship between IV and expected returns. The paper concludes with Section 8.

2. Motivation

Entropy has long been a key concept in information theory, as shown in works by Holm (1993) and Maasoumi (1993). Drawing from this foundation, our empirical study is inspired by various asset pricing research. The speculative tendencies of investors have spurred the creation of several theoretical models to understand the impact of these behaviors on asset prices. For instance, Usta and Kantar (2011), along with Caplin, Dean, and Leahy (2022), have proposed models that show a mix of investor preferences regarding entropy. In their models, some investors are drawn to entropy, while traditional investors focus on mean-variance optimization to maximize their portfolio returns. These investors assign a high value to stocks that exhibit positive entropy and often maintain portfolios that are not fully diversified, to enhance their exposure to positive entropy. In the market, prices reflect this, resulting in stocks with high entropy typically showing negative returns compared to the overall market portfolio. In a balanced market, where portfolio weights are carefully considered, the additional cost of acquiring more shares equals the benefit gained from them. As a result, in such a market equilibrium, investors often choose portfolios that are less diversified, leading to the pricing of total entropy and IE as described by Gulko (1999, 2002). Maasoumi and Racine (2002) have found that their model predicting expected entropy can effectively explain investor behaviors in stock markets.

An additional aspect of our research is the comparison between IE and other unique moments (characteristics) of stocks. This approach is inspired by Shannon's work in 1948, where he compared entropy with variance (a measure of volatility). Earlier studies like those by Maasoumi and Theil (1979) investigated entropy-based measures related to income differences, while Mukherjee and Ratnaparkhi (1986) and Cressie (1993) explored the link between entropy and volatility. Jiang, Wu, and Zhou (2018) introduced an entropy-focused method for assessing asymmetric movements in markets, finding that greater asymmetric comovement tends to predict higher stock returns. Chernov, Graveline, and Zviadadze (2018) employed entropy as a broad measure of variance in exchange rates to gauge currency risk. Entropy is particularly useful as it captures both regular and extreme risks in a single value, equating to variance under normal distribution but extending to include more complex risks otherwise. However, there has been limited research on how entropy relates to, or differs from, other statistical moments in stock analysis. Our study aims to illuminate this area by assessing whether IE and other unique stock characteristics are reflected in stock prices. We build upon the work of Ebrahimi, Maasoumi, and Soofi (1999) and Maasoumi and Racine (2002), focusing on comparing IE with other well-known idiosyncratic moments in stock returns.

In a different approach, Buchen and Michael created a model based on investors' prospect theory, showing that even though investors have varied holdings, the market balance they reach leads them to maintain portfolios that are not very diversified. The theory of cumulative prospect utility suggests that investors tend to give more importance to extreme probabilities. According to Gulko (1999, 2002), in situations where assets have confined return patterns, market balances formed under prospect theory can determine the price of entropy in those asset returns. Philippatos and Wilson (1972) discovered that having ideal expectations can lower the average returns of such confined assets in market balance. These theoretical findings, which connect entropy with expected returns, serve as the foundation for our empirical research.

Lastly, our study is inspired by the role of entropy in analyzing the predictability of financial markets over time. Researchers like Darbellay and Wuertz (2000) have explored the effectiveness of entropy in studying financial time series. Dimpfl and Peter (2018) demonstrated that investors could use group transfer entropy for predicting market volatility. Moreover, entropy has been used to measure market efficiency in various sectors, including foreign exchange (Oh, Kim, & Eom, 2007), stocks (Risso, 2008, 2009), and commodities (Martina, Rodriguez, Escarela-Perez, & Alvarez-Ramirez, 2011; Ortiz-Cruz, Rodriguez, Ibarra-Valdez, & Alvarez-Ramirez, 2012; Kristoufek & Vosvrda, 2014), focusing on how well an entire data series can be predicted using total entropy. However, entropy's ability to forecast events with small probabilities presents challenges. Current methods in asset pricing often rely on lagged predictors like IV, IS, and IK as risk indicators, assuming that these factors remain stable over time. Yet, this stability is questionable for IE. For instance, Maasoumi and Racine (2002) developed a model to forecast entropy and noticed that it somewhat reveals nonlinear relationships within stock return series. This issue encourages us to develop a new approach for estimating the expected entropy.

3. An entropy prediction model

We present a model predicting entropy, which incorporates historical returns, established idiosyncratic risk factors, and typical company traits. Our first step involves applying the well-known Fama and French (1993) three-factor model to daily total stock returns data for each company:

$$r_{i,d} = \alpha_i + \beta_{i,MKT} MKT_d + \beta_{i,SMB} SMB_d + \beta_{i,HML} HML_d + \varepsilon_{i,d}, \tag{1}$$

where $r_{i,d}$ represents the excess return of company i's stock on day d. MKT_d is the market's excess return on the same day. SMB_d indicates the difference in returns between portfolios of small-cap and large-cap stocks, while HML_d shows the difference in returns between stocks with high and low book-to-market ratios. The various β values are coefficients representing risk measures determined through regression analysis. Finally, $\varepsilon_{i,d}$ is the residual return of stock i on day d.

We utilize the residual returns calculated from equation (1) to create a measure of IE, based on fundamental concepts from information theory. Entropy quantifies the likelihood spread within a probability distribution. The commonly adopted method for this is Shannon entropy. However, extensions to Shannon's original theory have introduced different entropy metrics, such as the widely-used Renyi entropy proposed in 1961. According to Renyi's approach, consider stock X^* as a variable that can take various outcomes (represented by residual returns, symbolized as $\varepsilon_{i,d,t}$), such as $\varepsilon_{1,1,1}, \varepsilon_{1,2,1}, \varepsilon_{1,3,1}, \ldots, \varepsilon_{1,30,1}$, where $\varepsilon_{i,d,t}$ is the residual return of stock i on day d in month t. The corresponding probabilities are denoted by $p_{i,t} = p(X^* = \varepsilon_{i,d,t})$, with $0 \le p_{i,t} \le 1$ and the sum of probabilities $\sum_{d=1}^{n} p_{i,t}(\varepsilon_{i,d,t}) = 1$. Hence, we define a generalized discrete entropy function for the stock X^* as follows:

$$IE_{i,t}(X^*) = \frac{1}{1-\alpha}\log\left(\sum_{d=1}^{n} p_{i,t}^{\alpha}(\varepsilon_{i,d,t})\right) n = 1, 2, \ldots, 30\, days \tag{2}$$

where α represents the order of entropy, which must be greater than or equal to 0 but not equal to 1. The value of α signifies the importance given to each possible outcome: a lower α results in less emphasis on the more probable outcomes, and the opposite is true for higher values of α. The most commonly used values for α are 1 and 2. Additionally, the logarithm in the formula is based on 2.

An α value of 1 represents a special case within generalized entropy. Using Hôpital's rule [2], we can understand that as α approaches 1, H_α converges to what is known as Shannon entropy. However, directly inserting $\alpha = 1$ into equation (2) leads to a zero in the denominator, which is problematic. The logarithmic function used in equation (2) is designed to express the amount of information produced by a specific occurrence of a variable in terms of the logarithm of its probability. The information obtained from stock i in month t can be articulated as follows [3]:

$$IE_{i,t}(X^*) = -\log_2 p_{i,t}(\varepsilon_{i,d,t}), \tag{3}$$

The logarithm used in equation (3) is designed to calculate the information produced by a particular event of a variable, expressed as the logarithm of its occurrence probability. Considering a continuous probability distribution with a density function $f(x)$, we construct a density function specifically to define IE. When there are n returns for stock X^* with probabilities $p_{i,t}$, the average information gain for the stock is determined as follows:

$$IE_{i,t}(X^*) = -\sum_{d=1}^{n} p_{i,t} log\, p_{i,t}(\varepsilon_{i,d,t}), n = 1, 2, \ldots, 30\, days \tag{4}$$

Let's assume $p_{i,t}$ represents the likelihood of stock i's residual returns in month t. Define $n_{\varepsilon_{i,t}^+}$ and $n_{\varepsilon_{i,t}^-}$ as the counts of positive and negative residual returns for stock i in month t, respectively. Also, let $n_{\varepsilon_{i,t}}$ represent the total count of stock i's residual returns in that month. With these definitions, equation (4) can be rephrased as follows:

$$IE_{i,t}(X^*) = -\left[\frac{n_{\varepsilon_{i,t}^+}}{n_{\varepsilon_{i,t}}}\log\left(\frac{n_{\varepsilon_{i,t}^+}}{n_{\varepsilon_{i,t}}}\right) + \frac{n_{\varepsilon_{i,t}^-}}{n_{\varepsilon_{i,t}}}\log\left(\frac{n_{\varepsilon_{i,t}^-}}{n_{\varepsilon_{i,t}}}\right)\right] \tag{5}$$

where $IE_{i,t}(X^*)$ is always positive because $\log\left(\frac{n_{\varepsilon_{i,t}^+}}{n_{\varepsilon_{i,t}}}\right)$ and $\log\left(\frac{n_{\varepsilon_{i,t}^-}}{n_{\varepsilon_{i,t}}}\right)$ are always negative. Equation (5) shows how we construct our empirical exercises' monthly entropy of daily residual returns.

While we rely on equation (5) for calculating individual stock's IE, the method for computing IE for a portfolio differs and is crucial to understand conceptually. Consider an investment in two stocks, labeled 1 and 2, with their respective residual returns $\varepsilon_{1,d,t}$ and $\varepsilon_{2,d,t}$. These returns have associated probabilities $p_{1,t}$ for stock 1 and $p_{2,t}$ for stock 2, across $d = 1, 2, \ldots, n$ days and $t = 1, 2, \ldots, m$ months. The IE for the portfolio is derived from the combined distribution of $\varepsilon_{1,d,t}$ and $\varepsilon_{2,d,t}$, resulting in a joint IE calculated as follows:

$$IE_{P,t}(\varepsilon_{1,\mathrm{d,t}}, \varepsilon_{2,\mathrm{d,t}}) = -\sum_{d=1}^{n}\sum_{t=1}^{m} p_{i,t}(\varepsilon_{1,\mathrm{d,t}}, \varepsilon_{2,\mathrm{d,t}})\log_2\left[p_{i,t}(\varepsilon_{1,\mathrm{d,t}}, \varepsilon_{2,\mathrm{d,t}})\right], \tag{6}$$

where $p_{i,t}(\varepsilon_{1,d,t}, \varepsilon_{2,d,t})$ indicates the likelihood of experiencing residual returns on stocks 1 and 2 during month t, with 'P' representing the portfolio.

When dealing with two independent stock returns, the IE of the portfolio is simply the combined total of each stock's individual IE, expressed as

$$IE_{P,t}(\varepsilon_{1,\mathrm{d,t}}, \varepsilon_{2,\mathrm{d,t}}) = IE(\varepsilon_{1,\mathrm{d,t}}) + IE(\varepsilon_{2,\mathrm{d,t}}), \tag{7}$$

Based on the theoretical background provided, we apply Shannon entropy as outlined in Equation (5), along with the principles of entropy construction for portfolios, in carrying out our empirical analysis.

To draw comparisons between IE and established idiosyncratic risk metrics, we also calculate $IV_{i,t}$, $IS_{i,t}$, and $IK_{i,t}$ of stock i in month t using the following equations:

$$IV_{i,t} = \left(\frac{1}{n}\sum_{d=1}^{n} \varepsilon_{i,d,t}^{2}\right)^{1/2}, \tag{8}$$

$$IS_{i,t} = \frac{1}{n}\frac{\sum_{d=1}^{n} \varepsilon_{i,d,t}^{3}}{IV_{i,t}^{3}}, \tag{9}$$

$$IK_{i,t} = \frac{1}{n}\frac{\sum_{d=1}^{n} \varepsilon_{i,d,t}^{4}}{IV_{i,t}^{4}} - 3 \tag{10}$$

IE, as defined by Benedetto *et al.* (2016), evaluates the irregularities or disorders in residual returns. This metric assesses the unpredictability of residual returns, with a high IE indicating a high degree of randomness and uncertainty. In essence, returns with high IE are characterized by a lack of discernible patterns, rendering them almost completely random. On the other hand, low IE suggests a more deterministic nature of returns, where patterns and predictability are more evident (Kristoufek & Vosvrda, 2014). This concept of entropy differs significantly from IV and IS. IV, specifically, measures the dispersion or spread of residual returns around their mean, focusing solely on the extent to which these returns deviate from the average. In contrast, IS delves into the directional bias and extent of distribution in the residual returns. It quantifies the asymmetry in the distribution, indicating whether the returns are more likely to lean towards one direction over the other.

While IS captures the asymmetry in the distribution of residual returns, it is distinct from IE and IV in its focus on the direction and extent of this asymmetry. IS reflects the relative length of the tails in the PDF, providing insights into how much the distribution of returns skews away from the norm. For instance, a positive IS value indicates that the distribution of returns has a longer right tail, suggesting a greater likelihood of higher-than-average returns. Conversely, a negative IS value implies the opposite, with a longer left tail indicating a greater likelihood of lower-than-average returns. IK, on the other hand, is a measure that indicates the peak value of the PDF curve at the average value. It specifically addresses the thickness of the tails and the steepness of the distribution curve, offering insights into the likelihood of extreme returns. Unlike IE and IV, which focus on randomness and variance, respectively, IK provides a unique perspective on the extreme values in the distribution, highlighting the potential for outlier events in the return series.

4. Data

Our dataset includes stocks listed on DataStream from January 1988 to June 2019. We use daily stock returns to compute monthly risk factors. Following Ang *et al.* (2009) in terms of the starting point for data collection and analysis, we have compiled information on all active stocks across 23 developed countries, spanning from 1 January 1988 to 30 June 2019. The dataset encompasses 3,100 stocks, excluding those priced under 5 dollars and the bottom 5% in terms of market value. The broad duration of this dataset ensures a comprehensive analysis that encompasses various economic cycles, financial crises, and diverse risk conditions.

5. Analyzing cross-sectional distribution of firm-level risk factors

Figure 1 displays the cross-sectional distribution of $IE_{i,t}$, $IS_{i,t}$, $IK_{i,t}$, and $IV_{i,t}$ for stocks, constructed using 60-day periods from January 1988 to June 2019. The data are sourced from stocks listed on DataStream. All these risk factors exhibit variations over time, with IS showing particularly notable changes. IE demonstrates less fluctuation but experienced significant movements during the recent financial crisis (2007–2009), a trend also observed in the variations of IV. Panel A further highlights that the international stock market experienced periods of high entropy, particularly during 1990 and the financial crisis of 1997–1998 [4].

In our asset pricing evaluations, we need to determine the anticipated IE ($E_t[IE_{i,t+T}]$) for firm i over a 60-day period in month t, as described in equation (5). It's crucial that these expected entropy calculations are based on the information accessible to investors during month t. To realistically model how investors might view expected entropy, we initially conduct separate cross-sectional regressions at the close of each month t, as follows:

$$IE_{i,t} = \beta_{0,t} + \beta_{1,t} IE_{i,t-T} + \beta_{2,t} IS_{i,t-T} + \beta_{3,t} IK_{i,t-T} + \beta_{4,t} IV_{i,t-T} + \beta_{5,t} MAX_{i,t-T} + \beta_{6,t} MIN_{i,t-T} + \lambda_t' X_{i,t-T} + \varepsilon_{i,t}, \tag{11}$$

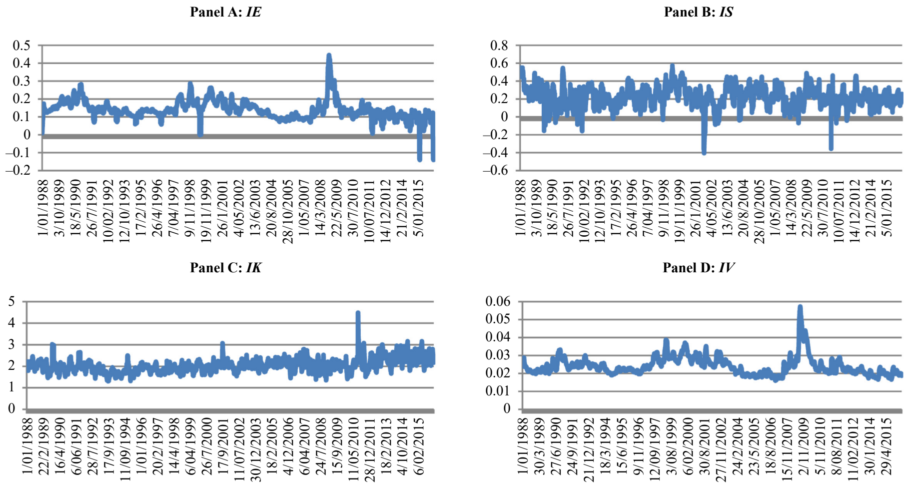


Figure 1.
Cross-sectional distribution of firm-level risk factors

Note(s): This figure presents the monthly cross-sectional distribution of IE (Panel A), IS (Panel B), IK (Panel C), and IV (Panel D) across 3,100 firms, spanning from January 1988 to June 2019

Source(s): Created by the author

where $X_{i,t-T}$ represents additional firm-specific risk factors for the month $t-T$. The time-based subscripts on the regression variables enable us to calculate them with the data available in month t. Equation (11) mirrors the approach used by Chen *et al.* (2001) and Boyer *et al.* (2010), but differs in that we include IE, IS, and IK. We also incorporate the MAX and MIN factors, as proposed by Bali *et al.* (2011), into Equation (11). The MAX factor involves the average of their highest daily returns from the previous month. Similarly, the MIN factor forms the average of the inverse of their lowest daily returns from the past month [5]. Using the regression coefficients from Equation (11) and the data available in month t, we calculate the expected IE for each firm i as follows:

$$E_t[IE_{i,t+T}] = \beta_{0,t} + \beta_{1,t}IE_{i,t} + \beta_{2,t}IS_{i,t} + \beta_{3,t}IK_{i,t} + \beta_{4,t}IV_{i,t} + \beta_{5,t}MAX_{i,t} + \beta_{6,t}MIN_{i,t} + \lambda_t' X_{i,t} \tag{12}$$

This approach enables us to observe how the relationship between firm-specific risk factors and IE varies over time, resulting in practical monthly estimates of expected IE.

We adopt this method to determine the expected IE over a 60-day formation period, though the selection of this period is somewhat subjective. This is based on the understanding that investors often focus on a stock's short-term growth prospects rather than long-term high returns. A 60-day formation period suggests that investors use data from the previous 60 days to make their estimates for Equations (11) and (12) [6].

The firm-specific risk factors $X_{i,t-T}$, as outlined in Equation (11), include market beta, the firm's book-to-market ratio, and size, following the model of Fama and French (1993). Additionally, it incorporates momentum (MOM) as described by Carhart (1997), which calculates the difference in returns between two portfolios with previously high returns and two with low prior returns. This also includes co-skewness as defined by Harvey and Siddique (2000), and liquidity as per Pastor and Stambaugh (2003). Utilizing the 60-day formation period, we conduct cross-sectional regression tests as per Equation (11) and calculate the expected entropy at the end of each month using Equation (12).

Table 1 presents the summary statistics for the risk factors used in our cross-sectional regression analyses. In Panel A, IE shows a relatively low mean value (0.14) compared to IS (0.22) and IK (2.01). This lower value of IE may indicate its effectiveness as a predictor, given that our aim is to forecast stock returns using this measure. Benedetto *et al.* (2016) observed that high entropy, indicative of significant irregularity, leads to the unpredictability of financial time series. Essentially, higher entropy suggests greater unpredictability, whereas lower entropy indicates fewer irregularities (disorders), enhancing the series' predictability. Martina *et al.* (2011) also found that higher entropy values are associated with more varied and less predictable market developments. Panels A and B also detail the descriptive statistics and correlations of these factors, respectively. IE shows the most positive correlation with IV (22%), MAX (17%), MIN (8%), co-skewness (11%), and HML (1%), though these correlations are relatively small. The strongest correlation observed is between IE and IV. It's noteworthy that other conventional risk factors in our sample exhibit either low positive or negative correlations, suggesting that IE is largely uncorrelated with these standard factors.

We recognize that Equation (11) is an economical model for predicting IE, and as such, it does not include some potential risk factors identified in previous studies. For example, Amaya *et al.* (2015) incorporate certain risk factors related to realized skewness. When we add these factors to our regression analyses, we observe an increase in explanatory power. However, including them results in the exclusion of many data points from our observations. We also tested our models with a leverage factor and found that, despite the lack of this data

	IE	*IS*	*IK*	*IV*	*MKT*	*SMB*	*HML*	*MOM*	Co-skew	Liquidity	MAX	MIN
Panel A: baseline results												
Mean	0.14	0.22	2.01	0.024	0.02	0.018	0.088	0.03	0.40	−0.09	1.28	−1.04
Median	0.13	0.22	1.99	0.023	0.05	0.011	0.082	0.06	0.40	−0.10	0.37	−0.39
Std. Dev	0.05	0.11	0.30	0.005	0.09	0.05	0.07	0.08	0.11	0.05	12.22	9.27
Skewness	1.06	−0.28	0.86	1.99	−0.21	8.57	−6.24	−1.03	0.27	−0.63	4.44	0.55
Kurtosis	8.83	4.02	5.38	9.94	11.47	205.08	111.4	16.91	2.16	36.38	8.26	1.34
Sharp ratio	2.80	2.00	6.70	4.80	0.22	0.36	1.25	0.37	3.63	−1.80	1.29	1.08
Panel B: correlations												
IE	1.00											
IS	−0.05	1.00										
IK	−0.21	0.05	1.00									
IV	0.22	−0.03	−0.22	1.00								
MKT	0.00	0.04	0.00	0.01	1.00							
SMB	−0.01	0.02	0.01	0.00	−0.14	1.00						
HML	0.01	−0.02	0.01	0.00	0.07	0.00	1.00					
MOM	−0.03	−0.02	0.00	−0.05	−0.22	0.05	−0.02	1.00				
Co-skew	0.11	0.03	−0.32	0.27	−0.01	−0.01	−0.02	0.02	1.00			
Liquidity	−0.26	−0.06	0.26	−0.11	0.00	0.01	0.02	−0.01	−0.64	1.00		
MAX	0.17	−0.12	−0.26	0.26	0.11	−0.09	−0.15	0.05	−0.10	−0.15	1.00	
MIN	0.08	−0.16	−0.29	0.28	0.09	−0.12	−0.09	0.01	−0.08	−0.12	−0.22	1.00

Note(s): Panel A provides descriptive statistics for the risk factors used to predict firm entropy through monthly cross-sectional regression analyses. IE, IV, IS, and IK are calculated using Equations (5), (8), (9), and (10), respectively, based on residuals determined by Equation (1). MKT represents the market's excess return; SMB denotes the excess return of small-cap stocks over large-cap stocks; HML is the excess return of high book-to-market stocks over low book-to-market stocks; MOM calculates the difference in returns between two portfolios with high previous returns and two with low previous returns. Co-skewness and liquidity are derived as per Harvey and Siddique (2000) and Pastor and Stambaugh (2003), respectively. MAX refers to the highest daily return within a month, and MIN is the inverse of the lowest daily return in the same period. Panel B additionally presents correlations among these risk factors. The dataset covers 3,100 firms from January 1988 to June 2019
Source(s): Created by the author

Table 1.
Descriptive statistics of *IE* prediction variables

for many firms in DataStream, it modestly enhances the adjusted R^2. Given these limitations, we choose to proceed with our sample using a more concise set of risk factors in our cross-sectional asset-pricing estimates, ensuring we maintain a broader range of firms in the analysis [7].

Panel A of Table 2 displays the outcomes of our monthly calculations based on Equation (11) for the primary sample set. Each row presents results from different regression models, ranging from model 1 to model 6. To summarize these regressions, we provide the average value of the coefficients and the percentage of months in which the estimated coefficients are significant at the 5% level and have the same sign as the average coefficients. However, as there's no adjustment for potential cross-sectional correlations in residuals, this significance should be considered only as a general indicator for comparison.

The table uses a 60-day period to define $IE_{i,t-T}$, $IS_{i,t-T}$, $IV_{i,t-T}$, and $IK_{i,t-T}$. Models 1 to 4 each use one of these factors to predict $IE_{i,t}$. In these models, $IE_{i,t-T}$, $IS_{i,t-T}$, and $IV_{i,t-T}$ are positively associated with $IE_{i,t}$ and show significant coefficients in 100%, 79%, and 94.2% of the monthly regressions, respectively. Conversely, $IK_{i,t}$ in model 4 negatively predicts $IE_{i,t}$ and is significant in 79.2% of the cases. Model 5, which employs all four factors together,

Table 2. The *IE* prediction regressions

Model	$\beta_{0,t}$	$IE_{i,t-T}$	$IS_{i,t-T}$	$IV_{i,t-T}$	$IK_{i,t-T}$	$MKT_{i,t-T}$	$SMB_{i,t-T}$	$HML_{i,t-T}$	$MOM_{i,t-T}$	$Coskew_{i,t-T}$	$LIQ_{i,t-T}$	$MAX_{i,t-T}$	$MIN_{i,t-T}$	$Adj.R^2$
Panel A: descriptive statistics														
1	0.039	0.62												7.01%
	(0.95)	(1.00)												
2	0.062		0.17											1.55%
	(0.79)		(0.79)											
3	0.035			6.88										38.15%
	(0.90)			(0.942)										
4	0.10				−0.038									3.99%
	(0.93)				(0.792)									
5	0.030	0.31	0.14	3.89	−0.007									44.29%
	(0.89)	(0.891)	(0.696)	(0.865)	(0.244)									
6	0.025	0.21	0.20	3.88	0.0011	−0.072	−0.016	0.012	−0.042	0.055	0.66	−0.059	0.012	50.11%
	(0.71)	(0.712)	(0.511)	(0.782)	(0.397)	(0.622)	(0.555)	(0.788)	(0.572)	(0.533)	(0.733)	(0.522)	(0.332)	
Panel B: robustness on different formation periods														
7 (30 days)	0.033	0.39	0.27	3.52	−0.010	−0.076	−0.022	0.024	−0.071	0.034	0.48	−0.068	0.016	61.03%
	(0.94)	(0.892)	(0.782)	(0.782)	(0.314)	(0.601)	(0.566)	(0.829)	(0.758)	(0.402)	(0.611)	(0.602)	(0.398)	
8 (90 days)	0.022	0.13	0.075	3.90	−0.004	−0.038	0.0096	0.012	−0.032	0.055	0.74	−0.033	0.010	35.11%
	(0.68)	(0.667)	(0.526)	(0.630)	(0.200)	(0.521)	(0.101)	(0.388)	(0.536)	(0.533)	(0.869)	(0.441)	(0.301)	
9	0.019	0.025	0.063	3.11	−0.005	−0.035	0.027	0.008	−0.022	0.083	0.84	−0.021	0.007	23.26%
(150 days)	(0.59)	(0.569)	(0.482)	(0.601)	(0.211)	(0.473)	(0.76)	(0.277)	(0.506)	(0.791)	(0.98)	(0.298)	(0.211)	
10	0.010	0.035	0.022	1.36	0.007	0.013	0.016	0.004	0.069	0.13	0.90	−0.015	0.005	13.29%
(365 days)	(0.44)	(0.501)	(0.273)	(0.324)	(0.38)	(0.282)	(0.49)	(0.112)	(0.322)	(0.977)	(1.00)	(0.112)	(0.109)	

Note(s): This table presents the average of risk coefficients over time, derived from monthly cross-sectional regression analyses as outlined in Equation (11). Panel A details regression results for 3,100 firms listed in Datastream, with all idiosyncratic risk measures calculated based on a 60-day period from January 1988 to June 2019. Panel B provides robustness checks for Equation (11), using various formation periods of 30, 90, 150, and 365 days for the idiosyncratic risk factors. Below each coefficient estimate, the table shows the percentage of coefficients that are statistically significant at the 5% level and have the same direction as the average coefficient
Source(s): Created by the author

yields similar findings. The results suggest that lagged IV is a stronger predictor of IE than lagged IS and IK. This might be due to the positive correlation between entropy and volatility, indicating that higher volatility leads to greater disorder in residual returns. Furthermore, as previous research (e.g. Fleming, Ostdiek, & Whaley, 1995; Busch, Christensen, & Nielsen, 2011) indicates that volatility can predict future volatility, it can also foresee future disorders. This correlation is also visible in Panels A and D of Figure 1. The adjusted R-squared values are highest in model 3 when using $IE_{i,t-T}$ and $IV_{i,t-T}$ individually. The coefficients in models 2 and 3 indicate that a standard deviation shock in $IV_{i,t-T}$ results in a sixfold change (6.88 times), while a one-skewness shock leads to less variation (0.17 times) in $IE_{i,t}$.

In Panel A, Model 6 incorporates all risk factors, with only $IK_{i,t-T}$ proving to be statistically insignificant. Higher values of $HML_{i,t-T}$, $LIQ_{i,t-T}$, and $Coskew_{i,t-T}$ are associated with higher values of $IE_{i,t}$, while increased values for $MKT_{i,t-T}$, $SMB_{i,t-T}$, and $MOM_{i,t-T}$ correspond to lower values of $IE_{i,t}$. The adjusted R-squared for the IE prediction regression sees an improvement when these additional risk factors are included in Model 6. The incorporation of these factors halves the predictive strength of both $IE_{i,t-T}$ and $IV_{i,t-T}$, while it raises the predictive capacity of $IS_{i,t-T}$ from 0.17 to 0.20. However, these modifications do not alter our univariate findings, where the estimated impact of a shock in $IV_{i,t-T}$ on $IE_{i,t}$ remains significantly higher than that of a shock in $IS_{i,t-T}$ on $IE_{i,t}$.

Panel B of Table 2 offers robustness checks for the predictive regression analyses presented in Panel A. In Model 7, a shorter 30-day period is used to calculate $IE_{i,t-T}$, $IS_{i,t-T}$, $IV_{i,t-T}$, and $IK_{i,t-T}$. This timeframe is selected in this subsection and Section 7 to facilitate comparisons with the findings of Ang *et al.* (2006, 2009). When juxtaposed with the baseline results of Model 6, the estimates in Model 7, which employs a shorter period for calculating risk factors, show greater significance and higher adjusted R-squared values. Nonetheless, the direction and relative sizes of the idiosyncratic risk coefficients remain consistent with those observed in Model 6.

In Panel B, Models 8, 9, and 10 replicate the regressions from Model 6. However, they calculate the metrics $IE_{i,t-T}$, $IS_{i,t-T}$, $IV_{i,t-T}$, and $IK_{i,t-T}$ using longer formation periods of 90, 150, and 365 days, respectively. While the size and statistical significance of the risk coefficients decrease over these extended periods, the results still align in terms of direction and significance with the foundational findings in Model 6.

Overall, the findings in this section indicate that a straightforward cross-sectional model, incorporating lagged IE, IS, IV, IK, market excess return, firm size, book-to-market ratio, momentum, co-skewness, liquidity, MAX, and MIN is effective in helping investors predict IE. The following section will delve into whether IE, as forecasted by this model, can elucidate the cross-section of expected returns.

6. Expected entropy and average returns

Our primary goal with the entropy prediction model is to determine if expected entropy can enhance our comprehension of the variations in stock returns. To achieve this, we conduct a series of conventional asset-pricing tests, assessing the relationship between expected entropy and average returns. This analysis is guided by the theoretical frameworks outlined in sections 1 and 2. Initially, we investigate how average returns vary among stocks with differing levels of expected entropy. Subsequently, we explore how predicted entropy impacts the variations in stock returns using the Fama and MacBeth (1973) approach.

6.1 Portfolios constructed by predicted entropy

Firstly, we calculate the expected entropy measures, $E_t[IE_{i,t+T}]$, at the end of each month from January 1988 to June 2019, following the methods described in Equations (11) and (12). These calculations use the risk factors from Model 6 in Table 2 for the cross-sectional regressions. After this, we categorize stocks into portfolios at the end of each month based on $E_t[IE_{i,t+T}]$ and compute the value-weighted returns for each portfolio in the following month (t+1). Similarly, we also sort stocks based on IS, IV, and IK for comparison.

Table 3 showcases descriptive statistics for the five portfolios arranged according to each idiosyncratic risk factor. Here, Portfolio 1 contains stocks with the lowest predicted risk, while Portfolio 5 contains stocks with the highest. The first column in each panel reveals the time-series average of the value-weighted portfolio returns. Notably, the average returns of the portfolios sorted by IE exhibit a consistent downward trend from Portfolios 1 to 5. The returns are significantly lower in Portfolio 5 (−1.33%) compared to Portfolio 1 (1.09%), resulting in a monthly spread of −2.42% with a *t*-statistic of −4.49. This indicates significant differences in average returns across stocks with varying levels of predicted entropy. The most substantial drop in mean returns occurs between the third and fourth portfolios.

Panels B and C indicate that portfolios organized according to IS and IV also display a consistently decreasing trend from Portfolio 1 to 5, mirroring the pattern seen in the IE-sorted portfolios. These findings align with the research of Ang *et al.* (2006) and Boyer *et al.* (2010), who observed similar trends for IV and IS. Conversely, Panel D reveals that portfolios sorted by IK show an increasing trend across the portfolios, corroborating the findings of Conrad *et al.* (2013). Given that existing research doesn't establish a clear expected relationship between total entropy (or IE) and subsequent returns, the congruence of our IV, IS, and IK results with prior studies lends credibility to our findings regarding IE.

While predicted entropy seems to be a crucial factor in determining returns, using lagged entropy alone isn't adequate for accurate return prediction. To further investigate this,

	Panel A: *IE*			Panel B: *IS*			Panel C: *IV*			Panel D: *IK*		
Portfolio	Mean	*IE*	Carhart alpha	Mean	*IS*	Carhart alpha	Mean	*IV*	Carhart alpha	Mean	*IK*	Carhart alpha
1 (Low)	1.09	0.06	1.19 (3.73)*	0.92	0.04	1.00 (3.40)*	1.02	0.011	0.98 (3.22)*	−1.20	0.82	−1.21 (−3.88)*
2	0.90	0.11	0.98 (2.33)*	−1.11	0.16	−1.02 (−2.80)*	−1.01	0.015	−1.00 (−3.01)*	−0.89	1.17	−1.17 (−3.35)*
3	0.69	0.12	0.74 (1.98)**	−1.14	0.18	−1.12 (−2.99)*	−1.08	0.019	−1.06 (−3.05)*	−0.92	1.47	−1.20 (−3.47)*
4	−1.08	0.17	−1.14 (−3.21)*	−1.19	0.26	−1.14 (−3.14)*	−1.14	0.021	−1.17 (−3.59)*	−1.00	2.00	−1.26 (−3.55)*
5 (High)	−1.33	0.31	−1.28 (−3.69)*	−1.20	0.40	−1.23 (−3.25)*	−1.20	0.039	−1.22 (−3.49)*	0.97	4.03	1.00 (2.37)*
5–1	−2.42 (−4.49)*		−2.47 (−4.46)*	−2.12 (−4.00)*		−2.23 (−4.06)*	−2.22 (−4.11)*		−2.10 (−4.10)*	2.17 (3.33)*		2.21 (3.44)*

Note(s): We calculate the estimates $IE_{i,t+T}$ (shown in Panel A) at the end of each month from January 1988 to June 2019, as described in Equations (11) and (12), using a 60-day formation period. The risk factors applied in these cross-sectional regressions are the same as those in Model 6 of Table 2. Subsequently, stocks are categorized into portfolios at the end of each month based on $E_t[IE_{i,t+T}]$, and we compute the value-weighted returns for each portfolio in the following month (t+1). Similar analyses are also performed for $IS_{i,t+T}$ (Panel B), $IV_{i,t+T}$ (Panel C), and $IK_{i,t+T}$ (Panel D). The panels in this table provide summary statistics for the five portfolios sorted by each risk factor. Portfolio 1 includes stocks with the lowest predicted risk, while Portfolio 5 comprises those with the highest. The first column in each panel shows the time-series average of the value-weighted portfolio returns, and the second column details their respective idiosyncratic risk. The third column in each panel presents the estimated alphas according to the Carhart (1997) four-factor model. Significance at 1% and 5% levels are indicated with * and **, respectively
Source(s): Created by the author

Table 3. Descriptive statistics of portfolios sorted into idiosyncratic risk measures

we conduct the same analysis as in Table 3 but categorize stocks into portfolios based on predicted entropy using $IE_{i,t-T}$ as the sole predictive factor (Model 1 of Table 2). This approach yields a marginal average return spread between portfolios 5 and 1 (-0.11%) [8], underscoring our conclusion that incorporating additional risk factors is essential for accurately estimating expected entropy in a meaningful way.

There are two key observations in this section. Firstly, the downward trends in the portfolios categorized by IE, IS, and IV, along with the upward trend in those sorted by IK, align with the findings in Table 2. In that table, IS and IV show positive relationships with IE, while IK displays a negative correlation. Secondly, and more importantly, the final column in each panel reveals that the return spreads for portfolios based on expected idiosyncratic risk are more substantial, even after risk adjustment. We also present the Carhart model alphas for each portfolio. The spread in alphas for IE (Panel A) is notably significant, with Portfolio 1 achieving a monthly alpha of 1.19% and Portfolio 5 achieving a monthly alpha of -1.28%, leading to a statistically significant spread of -2.47% per month. In essence, Table 3 illustrates that predicted entropy has a negative association with expected returns, even after accounting for standard risk factors.

6.2 FM *regressions*

In this subsection, we delve deeper into the pricing effects of IE by employing cross-sectional regression analyses based on the Fama and MacBeth (1973) (FM) methodology. Our analysis reveals a consistent, significant statistical and economic relationship between predicted entropy and average returns across different stocks, a trend that remains even when we account for standard risk factors.

To determine the expected entropy measures $E_t[IE_{i,t+T}]$, we use data from January 1988 to June 2019 and a 60-day formation period, as explained in Equations (11) and (12). The risk factors for these regressions are derived from Model 6 in Table 2. Each month, we categorize stocks into 100 portfolios based on their expected entropy and then calculate the value-weighted returns for these portfolios. Subsequently, we conducted the following cross-sectional regression for each month t:

$$r_{p,t+1} = \gamma_{0,t} + \gamma_{1,t} E_t[IE_{p,t+T}] + \phi_t' Z_{p,t} + \varepsilon_{p,t}, \tag{13}$$

where $r_{p,t+1}$ represents the value-weighted monthly return for portfolio p in the following month t+1; $E_t[IE_{p,t+T}]$ indicates the anticipated IE for portfolio p, calculated as the value-weighted average of firm-level IE for all stocks within portfolio p; and $Z_{p,t}$ is a vector representing the loadings of standard risk factors as control variables, which are determined at the end of each month t. The subscripts in the regression coefficients highlight that they are individually estimated for each month t in our dataset. The other risk factors (control variables) are the value-weighted averages of their firm-level equivalents.

In our analysis, $Z_{p,t}$ includes twelve additional risk factors. These factors are IV (value-weighted average firm-level volatility), IS (value-weighted average firm-level skewness), and IK (value-weighted average firm-level kurtosis), each computed for all stocks in portfolio p using formulas (8), (9), and (10), respectively, over a 60-day formation period. We also consider market excess return ($MKT_{p,t}$), size ($SMB_{p,t}$), book-to-market ratio ($HML_{p,t}$), and momentum ($MOM_{p,t}$), as defined in section 3 for month t. These are used to account for previously established relationships between these factors and expected returns, as they are integral to our entropy prediction model, Eq. (11). The coefficients for MKT, SMB, and HML are based on the Fama and French (1993) factors, while MOM uses the Carhart (1997) momentum factor. Furthermore, $Z_{p,t}$ includes the Pastor and Stambaugh (2003) liquidity factor ($LIQ_{p,t}$) and the Harvey and Siddique (2000) co-skewness ($Coskew_{p,t}$). It also represents MAX and MIN factors, as introduced by Bali *et al.* (2011), which are the averages of the highest and lowest daily returns, respectively, over the past two months. All these factors are calculated using daily data from day t-59 to the end of day t.

Table 4 presents the average over time of the γ and ϕ loadings and their associated t-statistics, calculated using Newey and West's (1987) standard errors. Model 1 showcases the cross-sectional pricing impact of expected entropy, revealing that the expected entropy coefficient is negative and significant at the 1% confidence level. Models 2 to 4 focus on the cross-sectional pricing of IS, IV, and IK, with both IS and IV coefficients being negative and significant, and the IK coefficient being positive and significant at the 1% confidence level. Models 5–7 include additional risk factor loadings as explanatory variables. The significance of the expected entropy coefficients is maintained whether we consider the expected entropy loading alone (Model 1) or alongside other factor loadings (Models 5–7). Since expected entropy is a composite of other risk factors, their inclusion in the regression reduces the significance of expected entropy. The other risk factor loadings' estimates generally align with expectations and mirror findings from Boyer *et al.* (2010).

Although there is no established risk price for entropy in existing literature for comparison, we focus on other risk factors. Consistent with expectations, the coefficients for IS and IV are negative and significant, while those for IK are positive and significant. Likewise, HML, co-skewness, and liquidity exhibit negative and significant loadings, whereas MOM shows insignificant negative loadings. In conclusion, Table 4 indicates that expected IE contributes to explaining the cross-sectional variation in expected returns, extending beyond the influence of traditional standard risk factors.

We also assess the effectiveness of the IE-based models in comparison to other models by looking at the R-squared values and the pricing errors. In columns 2 to 4 of Table 4, it is observed that the model using the IE portfolio exhibits higher R-squared values and lower pricing errors compared to the IV model, while it shows lower R-squared values and higher pricing errors when compared to the IS and IK models. Consequently, IE demonstrates better performance than IV in terms of achieving higher R-squared and lower pricing errors.

6.3 Robustness checks

To verify the consistency of the results presented in Table 4, we performed two robustness tests, the outcomes of which are detailed in Table 5. These tests involve altering the risk formation period and the number of portfolios to assess the stability of our findings related to entropy pricing. Through these checks, we consistently observe that the coefficients of predicted entropy maintain their negative and significant nature.

6.3.1 Alternative formation periods and portfolios. Table 4 previously presented the Fama-MacBeth (FM) regression results using a 60-day formation period, calculating the expected entropy measures $E_t[IE_{i,t+T}]$ as described in Equations (11) and (12). In this continuation, we replicate the analysis of Table 4 but apply the measures $E_t[IE_{i,t+T}]$ over different formation periods: 30, 90, 150, and 365 days. These expected IE measures were computed at the end of each month from January 1988 to June 2019. For estimating the expected IE, we used the risk factors from models 7 through 10 of Table 2 in the cross-sectional regressions. Panel A of Table 5 displays the outcomes for each formation period, showing that in every scenario, the expected IE coefficient remains negative and is statistically significant at the 1% confidence level.

Table 4 initially presented FM regression results based on monthly sorting of stocks into 100 portfolios by expected entropy. We now adjust the number of portfolios to both a smaller (50 portfolios) and a larger (200 portfolios) scale and rerun the cross-sectional tests on these newly sorted portfolios. Panel B of Table 5 replicates the analysis from Table 4 but uses 50 and 200 portfolios as test assets. The findings indicate that the FM regression executed with

Regressors	1	2	3	Models 4	5	6	7
Constant	0.12	0.11	0.12	0.13	0.13	0.20	0.21
	(1.00)	(1.79)***	(0.88)	(1.19)	(0.77)	(0.73)	(0.70)
$E_t[IE_{p,t+T}]$	−0.66				−0.66	−0.43	−0.33
	(−4.25)*				(−4.05)*	(−3.40)*	(−3.01)*
$IS_{p,t}$		−1.03			−0.98		−0.90
		(−8.24)*			(−7.79)*		(−6.94)*
$IV_{p,t}$			−0.22		−0.15		−0.11
			(−3.02)*		(−2.96)*		(−2.72)*
$IK_{p,t}$				5.22	5.22		3.44
				(15.17)*	(15.16)*		(12.34)*
$MKT_{p,t}$						−0.13	−0.10
						(−3.93)*	(−2.45)*
$SMB_{p,t}$						0.061	0.041
						(3.01)*	(1.92)***
$HML_{p,t}$						−0.11	−0.09
						(−3.66)*	(−3.33)*
$MOM_{p,t}$						−0.0022	−0.002
						(−0.44)	(−0.40)
$Coskew_{p,t}$						−0.16	−0.14
						(−4.22)*	(−3.96)*
$LIQ_{p,t}$						−0.93	−0.89
						(−6.18)*	(−6.12)*
$MAX_{p,t}$						−0.069	−0.064
						(−3.22)*	(−3.09)*
$MIN_{p,t}$						−0.009	−0.008
						(−1.11)	(−1.00)
χ^2	74.25	133.12	63.12	188.16	592.14	844.22	992.28
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
Pricing error	0.035	0.029	0.04	0.018	0.0086	0.0052	0.0033
AdjR^2	0.029	0.041	0.027	0.089	0.17	0.30	0.39

Note(s): This table outlines the results from the Fama and MacBeth (1973) regressions, along with their average coefficients, as per Equation (13). We analyze 100 portfolios that are sorted each month according to $E_t[IE_{p,t+1}]$. These regressions are calculated at the end of each month, from January 1988 to June 2019. For portfolio p, expected entropy and other risk factors are determined by the value-weighted average of firm-level measures across all stocks in the portfolio. The measures $E_t[IE_{p,t+1}]$ are computed as described in Equations (11) and (12) using a 60-day formation period. To estimate $E_t[IE_{p,t+1}]$, the cross-sectional regressions utilize the risk factors from Model 6 in Table 2. Using the same formation period, we also calculate $IV_{p,t}$, $IS_{p,t}$, and $IK_{p,t}$ for portfolio p via Equations (8), (9), and (10), respectively. $MKT_{p,t}$ represents the excess market return at month-end t, $SMB_{p,t}$ and $HML_{p,t}$ are the excess returns of small-cap stocks over large-cap stocks and high book-to-market stocks over low book-to-market stocks, respectively, in month t. $MOM_{p,t}$ is the difference in returns between two high prior return portfolios and two low prior return portfolios for that month. $Coskew_{p,t}$ and $LIQ_{p,t}$ correspond to Harvey and Siddique's (2000) co-skewness and Pastor and Stambaugh's (2003) liquidity measures, respectively, for month t. $MAX_{p,t}$ and $MIN_{p,t}$ are Bali, Cakici, and Whitelaw's (2011) maximum and minimum factors, representing the average of the highest and the inverse of the lowest daily returns over the past two months. The table includes average coefficients and Newey and West (1987) *t*-statistics (shown in parentheses), along with average adjusted-*R*-squared values. Significance at 1% and 10% levels are indicated with * and ***, respectively. The Fama-MacBeth *t*-statistics and the χ^2 test results are presented in parentheses and brackets, respectively

Source(s): Created by the author

Table 4.
FM regressions

200 portfolios displays greater statistical significance than that executed with 50 portfolios, as evidenced by higher *t*-statistics and adjusted *R*-squared values. This significance surpasses the baseline results shown in the last column of Table 4. Nonetheless, the IE

Regressors	30	Panel A: formation periods 90	150	365	Panel B: number of portfolios 50	200
Constant	0.21	0.22	0.22	0.20	0.19	0.20
	(0.72)	(0.79)	(0.84)	(0.63)	(0.70)	(0.69)
$E_t[IE_{p,t+T}]$	−0.70	−0.71	−0.67	−0.45	−0.33	−0.44
	(−5.66)*	(−5.87)*	(−5.15)*	(−4.92)*	(−4.08)*	(−4.72)*
$IS_{p,t}$	−0.71	−0.96	−0.39	−0.45	−0.89	−0.91
	(−6.35)*	(−7.12)*	(−3.68)*	(−3.76)*	(−6.66)*	(−6.55)*
$IV_{p,t}$	−0.39	−0.16	−0.13	−0.099	−0.09	−0.16
	(−3.44)*	(−2.83)*	(−2.81)*	(−2.18)*	(−1.93)***	(−2.84)*
$IK_{p,t}$	3.50	2.78	3.93	4.77	2.16	3.88
	(12.03)*	(9.74)*	(13.01)*	(17.88)*	(9.43)*	(13.12)*
$MKT_{p,t}$	−0.13	−0.12	−0.12	−0.11	−0.13	−0.27
	(−2.99)*	(−2.45)*	(−2.66)*	(−2.01)**	(−2.33)**	(−3.05)*
$SMB_{p,t}$	0.015	0.046	0.04	0.038	0.028	0.036
	(0.84)	(2.45)*	(2.14)**	(1.66)	(1.78)***	(1.77)***
$HML_{p,t}$	−0.18	−0.26	−0.24	−0.24	−0.11	−0.16
	(−3.22)*	(−3.33)*	(−3.22)*	(−3.44)*	(−2.53)*	(−2.78)*
$MOM_{p,t}$	−0.0036	−0.0012	−0.0028	−0.006	−0.0038	−0.016
	(−1.33)	(−0.41)	(−1.11)	(−2.54)*	(−0.49)	(−0.84)
$Coskew_{p,t}$	−0.12	−0.16	−0.19	−0.23	−0.11	−0.16
	(−3.99)*	(−3.94)*	(−3.93)*	(−4.08)*	(−2.36)*	(−4.01)*
$LIQ_{p,t}$	−0.80	−0.92	−0.95	−0.96	−0.79	−0.95
	(−5.43)*	(−5.99)*	(−6.04)*	(−5.88)*	(−5.34)*	(−5.96)*
$MAX_{p,t}$	−0.060	−0.067	−0.081	−0.084	−0.054	−0.079
	(−2.91)*	(−2.99)*	(−3.86)*	(−3.93)*	(−2.52)*	(−3.73)*
$MIN_{p,t}$	−0.007	−0.009	−0.011	−0.012	−0.006	−0.010
	(−0.88)	(−0.99)	(−1.12)	(−1.19)	(−0.69)	(−1.01)
χ^2	962.15	994.22	1014.06	1008.33	977.22	999.26
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
Pricing error	0.0036	0.0030	0.0026	0.0028	0.0034	0.0029
AdjR^2	0.34	0.38	0.44	0.43	0.36	0.41

Note(s): This table presents the results of the Fama and MacBeth (1973) regressions and their mean coefficients using equation (13). It utilizes monthly sorted portfolios based on $E_t[IE_{p,t+1}]$. In Panel A, columns 1 to 4 display the FM regression outcomes computed through measures $E_t[IE_{i,t+T}]$, in accordance with equations (11) and (12), for the formation periods of 30, 90, 150, and 365 days, respectively. We formulated the expected IE measures at each month's end from January 1988 to June 2019. The expected IE is estimated using the risk factors from models 7 to 10 in Table 2 for the cross-sectional regressions. Panel B's columns 1 (2) showcase stocks sorted into 50 (200) portfolios for every month, based on expected entropy. These cross-sectional regressions are calculated monthly for our sample, from January 1988 to June 2019. For portfolio p, expected entropy and other risk factors are the value-weighted average of firm-level factors across all stocks in the portfolio. The measures $E_t[IE_{p,t+1}]$, as detailed in equations (11) and (12), are computed using a 60-day formation period. Table 4 defines other risk factors. We present average coefficients and Newey and West (1987) t-statistics (in parentheses), alongside average adjusted-R2. Levels of significance are denoted as *, **, and *** for 1%, 5%, and 10%, respectively. The FM t-statistics are shown in parentheses, while the χ^2 test results are in brackets
Source(s): Created by the author

Table 5. Robustness checks on *FM* regressions

coefficients consistently remain negative and significant. The IV coefficient, however, shows insignificance (significance) in the 50-portfolio (100-portfolio) sample.

Additionally, we conduct two further checks in this section [9]. Firstly, we explore the impact of the risk formation period by calculating $IE_{i,t+T}$ using monthly instead of daily returns. Specifically, we apply 5- and 10-year formation periods of monthly returns to compute IE and incorporate these measures in our entropy-forecasting regressions and related cross-sectional pricing tests. The outcomes mirror the relationships and significance levels found in our baseline analysis, which used daily returns to construct entropy measures. For instance, the coefficient on $E_t[IE_{i,t+T}]$ is -0.36 with a t-statistic of -4.34, akin to column 7 of Table 4 but calculated using 5 years of monthly returns.

Lastly, we shift our focus to total entropy measures rather than IE measures for entropy forecasting and pricing tests. This adjustment is important to gauge our findings' reliance on the Fama and French three-factor model (Equation (1)) used for deriving IE measures. Our theoretical model suggests that investors are concerned with not just IE but the total entropy of their portfolio. When we conduct regressions using total entropy measures, there is a slight alteration in our pricing results, but the coefficient on $E_t[IE_{i,t+T}]$ remains significantly negative, at -0.72 with a t-statistic of -5.02, similar to column 7 of Table 4.

6.3.2 Alternative entropy factors. In Equation (5), the calculation of Shannon entropy is based on the assumption that α equals one, which is a common practice in constructing entropy measures. However, modifying α and incorporating additional assumptions into Shannon entropy necessitates verifying whether our findings remain valid under these changes. To confirm that our Shannon entropy isn't influenced by the monthly total return, we redo our empirical analyses using three alternative entropy measures. The first of these is the Renyi (1961) collision entropy measure, where α is set to two [10].

$$IE_{i,t}(X^*) = -\log \sum_{d=1}^{n} p_{i,t}^2(\varepsilon_{i,d,t}), \tag{14}$$

The second entropy measure we use is the Tsallis (1988) entropy, where for any positive real number α, the entropy of order α for a probability $p_{i,t}$ on a finite set X is defined as follows:

$$IE_{i,t}(p_{i,t}) = \begin{cases} \dfrac{1}{\propto - 1}\left(1 - \displaystyle\sum_{d\epsilon X} p_{i,t}^{\alpha}(\varepsilon_{i,d,t})\right), if \propto \neq 1 \\ -\displaystyle\sum_{d\epsilon X} p_{i,t} ln\, p_{i,t}(\varepsilon_{i,d,t}), if \propto = 1 \end{cases} \tag{15}$$

This measure is similar to Shannon entropy, but it differs in that the degree of homogeneity under the convex linearity condition is set to α rather than 1.

Lastly, we apply the Kullback and Leibler (1951) cross-entropy measure, which is based on two key assumptions: (1) each probability $p_{i,t}$ is greater than or equal to zero, and (2) the sum of all probabilities equals one. To fulfill these assumptions, it's necessary to measure the divergence between two probability distributions, namely $P = (p_{1,1}, p_{1,2}, \ldots, p_{1,12})$ and $Q = (q_{1,1}, q_{1,2}, \ldots, q_{1,12})$. The definition of this measure is as follows:

$$IE_{i,t}(P : Q) = \sum_{d=1}^{n} p_{i,t} ln \frac{p_{i,t}(\varepsilon_{i,d,t})}{q_{i,t}(\varepsilon_{i,d,t})}, \tag{16}$$

We create three alternative entropy factors by employing Equations (14), (15), and (16), and then conduct our empirical analyses again for each of these measures. In every instance, the outcomes closely align with those of Model 7 in Table 4. Across all regression models, we observed a consistently significant negative price for entropy risk. While Panel C indicates somewhat lower statistical significance compared to the other panels, the negative and significant nature of entropy risk is still evident.

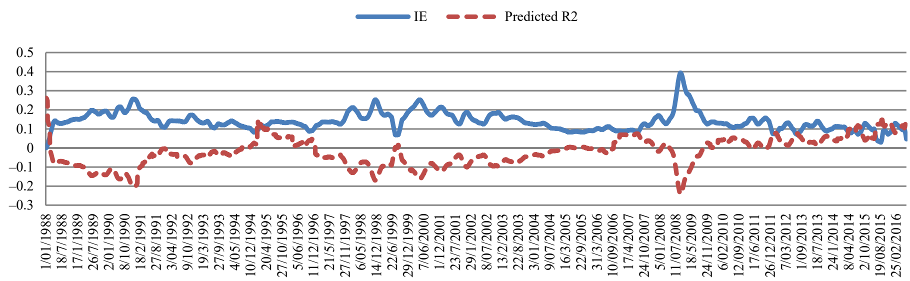
6.3.3 Time-series effects. Figure 1 illustrates the temporal characteristics of predicted entropy. We now turn our attention to addressing whether our cross-sectional pricing results exhibit any significant temporal variations. For each month t, Figure 2 presents a rolling average of the predicted entropy premium ($\gamma_{1,t}$) calculated from Equation (13) for the days $t-59$ to day t using our sample of 3,100 firms. Additionally, it includes a plot of the Adjusted R-squared estimated from Equation (11) for month t. The time-series graph displays patterns similar to those observed in Panel A of Figure 1, notably a disturbance during 1997–1998 and a more pronounced negative impact during 2007–2009. These fluctuations align with the speculative episodes reflected in the distribution of cross-sectional entropy shown in Panel A of Figure 1.

The series demonstrates a negative correlation with the graph of the predicted entropy premium. This suggests that the entropy premium tends to be more pronounced during speculative periods when expected entropy can be forecasted with higher accuracy and ease. To examine these relationships further, we proceed to conduct an analysis using Equation (17):

$$\overline{\gamma}_{1,t} = \delta_0 + \delta_1 \mu_{E_t[IE_{i,t+T}]} + \delta_2 \sigma_{E_t[IE_{i,t+T}]} + \delta_3 skew_{E_t[IE_{i,t+T}]} + \delta_2 kurt_{E_t[IE_{i,t+T}]} + \delta_3 R^2_{pred,t} + \varepsilon_t, \tag{17}$$

where $\mu_{E_t[IE_{i,t+T}]}$ represents the cross-sectional average of predicted entropy for month t; $\sigma_{E_t[IE_{i,t+T}]}$ is the cross-sectional standard deviation of predicted entropy in the same month; $skew_{E_t[IE_{i,t+T}]}$ denotes the cross-sectional skewness of predicted entropy; $kurt_{E_t[IE_{i,t+T}]}$ is the cross-sectional kurtosis of predicted entropy; and $R^2_{pred,t}$ is the adjusted R-squared from Equation (11). The estimated coefficients of Equation (17) and their standard errors are presented in Table 6. Adhering to methods from Pagan (1984) and Shanken (1992), we also adjust the standard errors for our estimated regressors.

Table 6 corroborates the suggestion from Panel A of Figure 1: the predicted entropy premium is most negative in times of high dispersion, high average, high skewness in predicted entropy, and when entropy is predictable. Conversely, this premium is most positive when the kurtosis of predicted entropy is high. The statistical significance of these five explanatory variables is notable. While these findings are preliminary and require further investigation to precisely define the temporal variations in entropy pricing, the evidence in Table 6 strongly supports the likelihood of such variations.



Note(s): This graph displays the 60-day moving average of the forecasted entropy premium γ_t, derived from equation (13) using our dataset of 3,100 firms. It includes the IE calculated based on equation (13) and the adjusted-R^2 from the cross-sectional regressions predicting entropy, as specified in equation (11)

Source(s): Created by the author

Figure 2. Predicted entropy premium and R^2

7. Entropy, skewness, volatility, and expected returns

Numerous studies in existing literature have focused on the relationship between IV and IS measures and expected returns, compared to the relatively less explored link between entropy and expected returns. Given our previous findings of a connection between IE and returns, this section aims to determine whether the relationship between IE and returns is directly tied to entropy itself or if it is influenced, at least in part, by IV and IS. To do this, we start by conducting the FM regressions as previously described and investigate how entropy might be associated with the established relationships between expected returns and the measures of IV and IS.

7.1 Regressions on individual stocks

The FM regressions described in subsection (6.2) calculate the price of IE by grouping stocks into quantile portfolios based on their expected entropy. However, this method might unintentionally exaggerate the impact of expected entropy and other risk factors, such as IS, IV, and IK. To mitigate this potential overestimation, an alternative approach is to apply the FM regressions at the individual stock level. While FM regressions are more commonly performed at the portfolio level in academic research, we opt to conduct our analyses at the individual stock level to assess if this approach influences our results. Consequently, we execute the following cross-sectional FM regression for each month:

$$r_{i,t+1} = \gamma_{0,t} + \gamma_{1,t} E_t[IE_{i,t+T}] + \phi_t' Z_{i,t} + \varepsilon_{i,t}, \tag{18}$$

where $r_{i,t+1}$ represents the daily return for firm i in the month following t, and the other risk factors are the same as those outlined in Equation (11). The key difference is that these factors are calculated for each individual stock rather than at the portfolio level.

Table 7 details the results of Equation (18). In Model 1, where only expected entropy is included in the regression, a significant negative coefficient is found at the 1% confidence level. Model 2, which includes only IS in the regression, reveals a larger coefficient for IS both in terms of economic significance (size) and statistical significance (*t*-statistic). Conversely, Model 3, which includes only IV, shows a negative coefficient for IV, but it has lower economic and statistical significance compared to the other two risk factors. Model 5 incorporates all three factors and echoes the findings of Models 1 to 3: the coefficient for expected IE is more significant both economically and statistically compared to IV, but less so than IS. Models 6 and 7 include expected entropy along with other risk factor loadings, and in both models, the coefficients for expected entropy remain negative and significant at the 1% level, with the economic and statistical significance of the IS (IV) coefficient being higher (lower) than that of IE.

Overall, while the risk coefficients in most models of Table 7 are smaller compared to our baseline portfolio results, the expected IE demonstrates considerable explanatory power in

Statistics	δ_0	$\mu_{E_t[IE_{i,t+T}]}$	$\sigma_{E_t[IE_{i,t+T}]}$	$skew_{E_t[IE_{i,t+T}]}$	$kurt_{E_t[IE_{i,t+T}]}$	$R^2_{pred,t}$
Estimations	0.31 (0.08)	−0.37 (−0.23)	−0.96 (−0.40)	−1.28 (−0.46)	2.09 (1.82)***	−1.79 (−0.74)
$N = 3100$ firms				$R^2 = 0.34$		

Note(s): This table presents the outcomes of Equation (17). To ensure accuracy, we have adjusted the standard errors of this model to consider the generated regressors as per Pagan (1984) and Shanken (1992), along with the time-series correlation methods outlined by Newey and West (1987). The symbol *** indicates that the results are statistically significant at the 10% level
Source(s): Created by the author

Table 6.
Time-series determinates of entropy pricing

Regressors	1	2	3	Models 4	5	6	7
Constant	0.12	0.10	0.12	0.11	0.13	0.22	0.22
	(1.10)	(3.66)*	(0.83)	(1.16)	(0.80)	(0.81)	(0.82)
$E_t[IE_{i,t+T}]$	−0.40				−0.38	−0.36	−0.30
	(−5.22)*				(−4.98)*	(−5.25)*	(−4.42)*
$IS_{i,t}$		−1.20			−1.15		−0.88
		(−7.96)*			(−7.66)*		(−9.88)*
$IV_{i,t}$			−0.30		−0.12		−0.44
			(−3.71)*		(−3.00)*		(−4.32)*
$IK_{i,t}$				3.44	3.26		2.80
				(11.08)*	(10.79)*		(11.99)*
$MKT_{i,t}$						−0.096	−0.098
						(−2.42)*	(−2.34)*
$SMB_{i,t}$						0.07	0.027
						(3.66)*	(1.89)
$HML_{i,t}$						−0.16	−0.13
						(−3.62)*	(−3.45)*
$MOM_{i,t}$						−0.006	−0.005
						(−2.24)**	(−1.98)**
$Coskew_{i,t}$						−0.13	−0.11
						(−3.93)*	(−3.77)*
$LIQ_{i,t}$						−0.91	−0.86
						(−5.32)*	(−5.04)*
$MAX_{p,t}$						−0.07	−0.067
						(−3.42)*	(−3.74)*
$MIN_{p,t}$						−0.01	−0.009
						(−1.33)	(−1.12)
χ^2	515.10	577.22	126.22	98.24	600.25	766.08	892.14
	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
Pricing error	0.0079	0.0069	0.029	0.039	0.0065	0.0054	0.004
AdjR^2	0.09	0.15	0.070	0.052	0.22	0.35	0.43

Note(s): This table displays the results of the Fama and MacBeth (1973) regressions, including their average coefficients as per Equation (18). These regressions are performed at the end of each month for the period from January 1988 to June 2019, using our sample. The measures $E_t[IE_{i,t+1}]$ are computed according to Equations (11) and (12) and are based on a 60-day formation period. For estimating $E_t[IE_{i,t+1}]$, we use the risk factors from Model 6 in Table 2. All risk factors are detailed in Table 2, with portfolio characteristics being the value-weighted averages of firm-level data, calculated using daily information from the 59th day before the end of month t to the end of day t. The table includes average coefficients and Newey and West (1987) t-statistics (shown in parentheses), as well as average adjusted R-squared values. The symbols * and ** represent statistical significance at the 1% and 5% levels, respectively. The Fama-MacBeth t-statistics and the results of the χ^2 test are indicated in parentheses and brackets, respectively
Source(s): Created by the author

Table 7. *FM* regressions on individual stocks

the individual stock analyses. When other risk factors are not included, expected entropy has a more explanatory power than IV. However, IV's explanatory power increases after adjusting for other risk factors. Given that expected entropy is a combination of other risk factors used as control variables, its explanatory power understandably diminishes when all risk factors are included in the regression. Table 7 also indicates that the explanatory power of expected entropy surpasses that of IV. These FM regression results confirm that our previous findings, which involved sorting stocks based on expected entropy, are not disproportionately influenced by this sorting method.

In Table 7, Models 1 to 4 show that when individual stocks are used as test assets, the IE-based model yields higher R-squared values and lower pricing errors compared to the IV and IK models, while it has lower R-squared values and higher pricing errors when compared to the IS models. This indicates that the IE model surpasses both IV and IK in terms of achieving higher R-squared and lower pricing errors.

7.2 Regressions with IS- *and* IV-*sorted portfolios*

This subsection examines whether categorizing stocks into portfolios based on expected entropy skews our results towards favoring IE over IS and IV. Initially, we group stocks into quantile portfolios based on IS as compared to expected IE and perform similar groupings for IV relative to expected IE. Subsequently, we utilize the Fama-MacBeth (FM) regressions to assess the risk factors' prices. If the sorting method biases our results towards the factor used for creating the quantile portfolios, then these FM regressions could offer a more accurate estimation of the impact of the expected IE factor.

In conducting the FM regressions, as specified in Equation (11), we differ only in that we sort stocks into 100 portfolios each month based on IV, rather than expected entropy. The outcomes of these regressions are detailed in Panels A and B of Table 8. When sorting by IV (Panel A), it's observed that the IE coefficients exhibit greater economic and statistical significance – evidenced by larger sizes, higher t-statistics, and increased adjusted R-squared values – compared to both IV and IS. Sorting by IS yields results similar to those in Table 4: the IE coefficients are negative and significant at the 1% level, and their economic and statistical significance are higher (lower) than IV (IS), as indicated by their sizes, t-statistics, and adjusted R-squared values.

In summary, Table 8 demonstrates that the notable explanatory power of expected IE is not influenced by the sorting methodology used, and that expected entropy retains its distinct explanatory power compared to IV and IS, even when these latter factors are used as the basis for portfolio formation.

7.3 Entropy and the IV *puzzle*

While earlier sections highlighted the pricing impacts of IE, we now aim to explore IE's role in explaining the negative relationship between IV and expected returns, as identified by Ang *et al.* (2006). Additionally, we seek to differentiate the influences of IE and IV on expected returns. This investigation is crucial for two key reasons. Firstly, a negative correlation between return and risk challenges conventional beliefs about investors' utility. Secondly, it brings attention to potential market inefficiencies, such as limited information disclosure or restrictions on short-selling, as noted by Boehme, Danielsen, Kumar, and Sorescu (2009) and Jiang, Xu, and Yao (2009). However, the IV puzzle, when viewed through the lens of entropy preference, is unlikely to stem from market imperfections. If investors favor stocks with positive entropy, they might be willing to accept lower returns on high IV stocks if those stocks promise high returns. This rationale aligns with the approach described in section 3 for calculating entropy, where entropy is inherently non-negative.

To delve into how predicted entropy affects the IV puzzle, we start by revisiting the principal findings of Ang *et al.* (2006) and then assess how these findings are altered when factoring in predicted entropy. Subsequently, we differentiate the effects of IE and IV and conduct a reverse analysis to determine the extent to which IV can account for our observations regarding the relationship between expected IE and returns.

7.3.1 IV *and expected returns.* In this section, we adopt Ang *et al.*'s (2006) methodology and start by categorizing stocks into quintile portfolios each month based on their $IV_{i,t}$, as detailed in Tables 1 and 2 We create these IV-based portfolios and determine their value-weighted returns for the subsequent month (t+1). For calculating $IV_{i,t}$, we use a 30-day

Table 8. *FM* regressions on *IS*- and *IV*-sorted portfolios

Regressors	Panel A: *IV* 1	2	3	4	5	6	7	Panel B: *IS* 1	2	3	4	5	6	7
Constant	0.12	0.10	0.11	0.11	0.13	0.21	0.22	0.11	0.09	0.09	0.11	0.12	0.20	0.20
	(0.98)	(2.50)*	(1.02)	(1.15)	(1.07)	(0.84)	(0.80)	(1.22)	(1.72)	(1.35)	(1.95)***	(0.48)	(0.72)	(0.63)
$E_t[IE_{p,t+T}]$	−0.68				−0.89	−0.66	−0.68	−0.62				−0.68	−0.60	−0.45
	(−5.33)*				(−5.52)*	(−5.36)*	(−5.52)*	(−4.66)*				(−5.35)*	(−5.67)*	(−4.83)*
$IS_{p,t}$		−0.42			−0.13		−0.40		−0.92			−0.48		−0.45
		(−3.12)*			(−1.32)		(−3.72)*		(−6.12)*			(−3.3)*		(−3.66)*
$IV_{p,t}$			−0.38		−0.32		−0.13			−0.32		−0.11		−0.095
			(−2.9)*		(−3.77)*		(−3.74)*			(−3.40)*		(−2.52)*		(−1.98)**
$IK_{p,t}$				6.35	5.14		4.01				5.99	5.01		4.73
				(6.34)*	(5.33)*		(4.33)*				(5.55)*	(3.89)*		(3.26)*
$MKT_{p,t}$						−0.13	−0.13						−0.13	−0.12
						(−2.41)*	(−2.61)*						(−3.06)*	(−2.2)**
$SMB_{p,t}$						0.048	0.039						0.049	0.04
						(2.62)*	(2.12)**						(2.55)*	(1.65)
$HML_{p,t}$						−0.18	−0.24						−0.25	−0.20
						(−4.94)*	(−6.88)*						(−3.32)*	(−3.24)*
$MOM_{p,t}$						−0.005	−0.003						−0.003	−0.006
						(−1.99)**	(−1.26)						(−1.22)	(−2.53)*
$LEV_{p,t}$						−0.11	−0.11						−0.13	−0.11
						(−3.14)*	(−3.14)*						(−3.15)*	(−3.21)*
$LIQ_{p,t}$						−1.05	−0.97						−0.98	−0.99
						(−5.68)*	(−5.06)*						(−5.92)*	(−5.99)*
$MAX_{p,t}$						−0.064	−0.06						−0.06	−0.05
						(−3.12)*	(−3.01)*						(−3.21)*	(−3.01)*
$MIN_{p,t}$						−0.008	−0.007						−0.007	−0.006
						(−1.05)	(−0.95)						(−0.98)	(−0.89)
χ^2	555.10	31.25	14018	510.22	615.30	809.02	921.29	498.44	52.14	86.14	500.29	555.21	762.16	894.15
	[0.00]	[0.04]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]	[0.03]	[0.00]	[0.00]	[0.00]	[0.00]	[0.00]
Pricing error	0.0092	0.078	0.033	0.0088	0.0071	0.005	0.0035	0.0080	0.057	0.044	0.009	0.0086	0.0062	0.0041
AdjR^2	0.14	0.0028	0.056	0.11	0.21	0.40	0.45	0.11	0.013	0.043	0.10	0.18	0.35	0.43

Note(s): This analysis follows the same methodology as in Table 4, with the only difference being that we now organize stocks into 100 portfolios each month based on IV. The outcomes of these IV-based pricing regressions are presented in Panel A. Following this, we apply the same categorization process using IS and display the results of the IS-based pricing regressions in Panel B. The symbols *, **, and *** indicate statistical significance at the 1%, 5%, and 10% levels, respectively. The Fama-MacBeth (FM) t-statistics and the results of the χ^2 test are provided in parentheses and brackets, respectively

Source(s): Created by the author

formation period, aligning with the approach of Ang *et al.* (2006, 2009). Additionally, we apply a 60-day formation period for other idiosyncratic risk measures, as per our initial analysis. Table 9 shows descriptive statistics for the returns of these portfolios, exhibiting patterns similar to those Ang *et al.* (2006) reported. The first two and last two columns of Table 9, which include average returns, return standard deviations, and CAPM and Fama and French (1993) three-factor model (FF) alphas for month t+1, are directly comparable to Ang *et al.* (2006)'s Table 10, Panel B. Notably, the high IV portfolio underperforms, with average monthly returns of −1.29%, while the low IV portfolio achieves 1.18%. The CAPM and FF model alphas also show significant spreads, with the FF alpha spread for portfolio 5–1 being −2.47% per month. This confirms Ang *et al.* (2006)'s observation of the IV puzzle being most evident in the high IV portfolio.

Table 9 also includes two entropy risk factors. The first, shown in column 4, is the total entropy time-series estimate for each portfolio's returns, displaying a trend where higher IV portfolios exhibit higher total entropy returns. The second factor, the firm entropy presented in column 6, represents the time-averaged value-weighted cross-sectional average of $IE_{i,t}$ for each portfolio, calculated using Equation (5) with a 60-day formation period. This indicates a strong link between IV and IE, as higher volatility portfolios tend to have higher firm entropy. These entropy factors suggest that lagged IV relates to portfolio return entropy [11]. The most significant increase in entropy, as shown in the mean returns, occurs between the second and third portfolios. Columns 5 and 9 provide the value-weighted cross-sectional averages of $IV_{i,t}$, $IE_{i,t}$, $IS_{i,t}$, $IK_{i,t}$, and $ln\ (Size_{i,t})$ within each portfolio. The relationships among IV, IE, and size support the idea of an entropy-preferring investor who speculates in smaller

Portfolio	1 Mean	2 Std. Dev	3 Skew	4 Entropy	5 *IV*	6 *IE*	7 *IS*	8 *IK*	9 Size	10 CAPM alpha	11 *FF* alpha
1 (Low)	1.18	0.012	0.013	0.12	0.014	0.07	0.11	2.89	12.88	1.24 (4.82)*	1.11 (4.35)*
2	1.00	0.012	0.015	0.15	0.019	0.09	0.17	1.90	12.24	0.33 (2.47)*	0.29 (2.52)*
3	−1.02	0.012	0.031	0.17	0.14	0.12	0.23	1.68	11.88	−0.04 (−1.88)***	−0.03 (−1.85)***
4	−1.06	0.012	0.24	0.22	0.28	0.17	0.28	1.82	10.99	−1.22 (−2.93)*	−1.15 (−2.89)*
5 (High)	−1.29	0.011	0.28	0.34	0.40	0.28	0.34	1.64	8.22	−1.42 (−3.77)*	−1.36 (−3.38)*
5–1	−2.47 (−4.45)*									−2.66 (−4.30)*	−2.47 (−4.56)*

Note(s): This table displays descriptive statistics for portfolios categorized by IV. We calculate the $IV_{i,t}$ estimates each month from January 1988 to June 2019 following the method in Equation (8). Stocks are then grouped into portfolios at the end of each month based on $IV_{i,t}$, and we calculate the value-weighted returns for each portfolio in the subsequent month (t+1). While all idiosyncratic risk factors are determined using a 60-day formation period, IV is computed over a 30-day period to allow for comparisons with the IV used by Ang *et al.* (2006, 2009). The portfolios are divided into five groups, with Portfolio 1 containing stocks with the lowest $IV_{i,t}$ levels and Portfolio 5 with the highest. Column 1 shows the average value-weighted returns of these portfolios over time. Columns 2 to 4 detail the time-series standard deviation, skewness, and total entropy of the portfolio returns. Columns 5 to 9 provide the value-weighted cross-sectional averages of $IV_{i,t}$, $IE_{i,t}$, $IS_{i,t}$, $IK_{i,t}$, and $Size_{i,t}$ within each portfolio. Columns 10 and 11 present the alphas and Newey and West (1987) *t*-statistics (in parentheses) for the CAPM and Fama-French's (1993) three-factor models. The symbols * and *** indicate statistical significance at the 1% and 10% levels, respectively
Source(s): Created by the author

Table 9.
Descriptive statistics of portfolios sorted into *IV*

Table 10. Robustness checks: *FM* regressions on alternative entropy measures

Regressors	Panel A: Renyi's entropy	Panel B: Tsallis's entropy	Panel C: Kullback and Leibler's entropy
Constant	0.13 (0.59)	0.15 (0.57)	0.07 (0.27)
$E_t[IE_{p,t+T}]$	−0.35 (−3.08)*	−0.41 (−3.66)*	−0.16 (−2.74)*
$IS_{p,t}$	−0.76 (−5.77)*	−0.83 (−5.98)*	−0.58 (−3.87)*
$IV_{p,t}$	−0.12 (−2.72)*	−0.27 (−2.93)	−0.08 (−1.77)***
$IK_{p,t}$	3.11 (12.74)*	3.35 (12.66)*	1.80 (5.00)*
$MKT_{p,t}$	−0.18 (−2.44)**	−0.35 (−3.98)*	−0.048 (−1.91)***
$SMB_{p,t}$	0.031 (1.54)	0.046 (1.55)	0.006 (0.94)
$HML_{p,t}$	−0.12 (−2.22)**	−0.27 (−3.82)*	−0.06 (−1.32)
$MOM_{p,t}$	−0.011 (−0.78)	−0.017 (−0.77)	−0.002 (−0.62)
$Coskew_{p,t}$	−0.12 (−3.66)*	−0.23 (−3.93)*	−0.06 (−2.23)*
$LIQ_{p,t}$	−0.80 (−4.53)*	−0.97 (−5.43)*	−0.39 (−2.77)*
$MAX_{p,t}$	−0.077 (−3.55)*	−0.082 (−3.63)*	−0.065 (−2.97)*
$MIN_{p,t}$	−0.010 (−1.08)	−0.011 (−1.15)	−0.008 (−0.85)
χ^2	990.22 [0.00]	997.21 [0.00]	600.10 [0.00]
Pricing error	0.0036	0.0033	0.0083
AdjR^2	0.37	0.39	0.17

Note(s): The methodology used here mirrors that of Table 4, with the only difference being the substitution of Shannon entropy (calculated using Equation (5) with Renyi's collision entropy (Equation (14), Tsallis's entropy (Equation (15), and Kullback and Leibler's entropy (Equation (16)). We then replicate the same pricing analyses as those shown in Table 4. The symbols *, **, and *** represent statistical significance at the 1%, 5%, and 10% levels, respectively. The Fama-MacBeth (FM) t-statistics and the χ^2 test results are noted in parentheses and brackets, respectively
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firms with highly unpredictable returns and is willing to accept lower average returns for the chance of substantial gains.

7.3.2 Conditional sorting. In this section, we explore the contribution of IE in clarifying the IV anomaly. Following the methodology used by Ang *et al.* (2006, 2009), we perform a double-sorting technique that incorporates expected entropy. At the end of each month, we first sort stocks into five quintiles based on their $E_t[IE_{i,t+T}]$, calculated as per Equations (11) and (12) using a 60-day formation period. The risk factors for Equation (11) are taken from Model 6 in Table 2. Within each $E_t[IE_{i,t+T}]$ quintile, stocks are further divided into five groups based on their IV. This approach creates 25 quantile portfolios. Subsequently, we calculate the value-weighted returns of these portfolios in the ensuing month (t+1), thereby considering the effect of expected entropy. The results of these sorting exercises are detailed in Table 11.

Panel A of Table 11 suggests that entropy preference plays a key role in the returns of IV portfolios. Column 1 shows that the range of average returns across portfolios sorted by conditional IV is significantly smaller than those sorted by unconditional IV. The highest-to-lowest IV portfolio yields a monthly return premium of −2.09% with a t-statistic of −3.08, which is lower than the premium reported in Table 9. This indicates that the IV puzzle identified by Ang *et al.* (2006) is considerably reduced when we account for expected IE.

Columns 2 to 4 of Panel A in Table 11 detail the standard deviation, skewness, and total entropy of returns for the portfolios ranked by IV, taking expected IE into account. Columns 5 to 9 provide value-weighted cross-sectional averages of IV, IE, and size, as well as the CAPM and Fama-French (FF) model alphas for each portfolio. Our findings indicate minimal variation in entropy across the five IV portfolios, suggesting that our expected entropy model accurately predicts entropy. The average firm entropy also shows smaller variations compared to the unconditional IV-sorted portfolios in Table 9, especially with slightly lower firm entropy in high IV portfolios.

Portfolio	1 Mean	2 Std. Dev	3 Skew	4 Entropy	5 *IV*	6 *IE*	7 Size	8 CAPM alpha	9 *FF* alpha
Panel A: double-sorted IV portfolios									
1 (Low)	0.84	0.010	−0.018	0.20	0.17	0.063	12.85	0.87 (3.48)*	0.90 (3.30)*
2	0.79	0.020	0.042	0.16	0.50	0.052	12.22	0.80 (3.16)*	0.80 (3.15)*
3	−1.00	0.035	0.079	0.13	0.59	0.045	9.44	−1.34 (−2.22)**	−1.30 (−2.12)**
4	−1.23	0.044	0.17	0.11	0.62	0.036	9.23	−1.41 (−2.92)*	−1.32 (−2.77)*
5 (High)	−1.25	0.112	0.26	0.08	0.67	0.031	7.33	−1.50 (−2.96)*	−1.24 (−2.48)*
5–1	−2.09 (−3.08)*							−2.37 (−3.22)*	−2.14 (−3.16)*
UMC	−0.32 (−2.25)**							−0.20 (−2.15)**	−0.23 (−2.26)*
Panel B: double-sorted IE portfolios									
1 (Low)	0.86	0.012	0.11	0.075	0.40	0.044	12.48	0.88 (3.18)*	0.90 (3.23)*
2	0.82	0.020	0.09	0.14	0.50	0.049	11.77	0.74 (3.16)*	0.60 (2.89)*
3	−0.98	0.078	0.081	0.17	0.52	0.054	10.34	−1.23 (−2.11)**	−1.06 (−1.78)***
4	−1.18	0.083	0.071	0.23	0.64	0.058	9.09	−1.28 (−2.18)**	−1.19 (−1.96)***
5 (High)	−1.30	0.090	0.109	0.27	0.72	0.066	7.66	−1.38 (−2.27)**	−1.35 (−2.01)**
5–1	−2.16 (−3.20)*							−2.26 (−3.26)*	−2.25 (−3.22)*
UMC	−0.28 (−2.22)**							−0.20 (−2.15)**	−0.23 (−2.26)*

Note(s): This table displays detailed statistics for portfolios arranged by expected entropy and IV. Each month, from January 1988 to June 2019, we calculate the $E_t[IE_{i,t+T}]$ estimates using a 60-day formation period as detailed in Equations (11) and (12). $IV_{P,t}$ is computed using a 30-day period as per Equation (8), allowing for comparison with Ang *et al.*'s (2006, 2009) IV analysis. Panel A shows the outcomes of sorting stocks into portfolios at each month's end based on $E_t[IE_{i,t+T}]$. Within each $E_t[IE_{i,t+T}]$ quintile, stocks are further divided into $IV_{i,t}$ quintiles, and we calculate the value-weighted return for these 25 portfolios in the subsequent month (t+1). We then compute the value-weighted average across each of the five $E_t[IE_{i,t+T}]$ quintiles to account for the impact of expected IE. Panel B reverses the roles of $E_t[IE_{i,t+T}]$ and $IV_{i,t}$ creating portfolios ranked by $E_t[IE_{i,t+T}]$ to control for the effects of $IV_{i,t}$. This panel also offers descriptive statistics for the five conditionally ranked portfolios, where Portfolio 1 contains stocks with the lowest $IV_{i,t}$ (Panel A) or $E_t[IE_{i,t+T}]$ (Panel B), and Portfolio 5 includes those with the highest risk. Column 1 presents the average value-weighted returns of these portfolios over time. Columns 2 to 4 detail the standard deviation, skewness, and total entropy of the portfolio returns. Columns 5 to 7 provide the value-weighted cross-sectional averages of $IV_{i,t}$, $IE_{i,t+T}$, and $Size_{i,t}$ within each portfolio. Columns 8 and 9 show the alphas and Newey and West (1987) *t*-statistics (in parentheses) for the CAPM and Fama-French (1993) three-factor models. The symbols *, **, and *** indicate statistical significance at the 1%, 5%, and 10% levels, respectively
Source(s): Created by the author

Table 11. Descriptive statistics of double-sorted portfolios

Columns 8 and 9 of Panel A report risk-adjusted pricing spreads across the portfolios using CAPM and FF model alphas. The highest IV portfolio exhibits a significant CAPM alpha of −2.47%, albeit less than the unconditional high IV portfolio from Table 9, indicating the persistence of the IV puzzle based on CAPM risk-adjusted alpha. For the FF model, the high

IV portfolio's alpha is −2.14%, which is also smaller than the unconditional high IV portfolio's FF alpha reported in Table 9. Thus, controlling for IE results in much smaller spreads in the CAPM and FF alphas.

The bottom row of Panel A uses an Unconditional Minus Conditional (UMC) portfolio to assess improvements in entropy forecasting. This UMC portfolio strategy involves taking long positions in the unconditional high/low IV portfolio and short positions in the conditional high/low IV portfolio. The results strongly support the explanatory power of the predicted entropy, with the CAPM alpha on the UMC portfolio being −0.20% (*t*-statistic of −2.15) and the FF alpha being −0.23% (*t*-statistic of −2.26).

Panel B of Table 11 follows a similar analysis as Panel A, but reverses the roles of expected entropy and IV. This approach aims to determine if IV contributes to explaining the returns of portfolios ranked by expected entropy, as shown in Table 3. The findings reveal that including IV reduces the return spread of the highest-to-lowest entropy portfolio compared to Table 9. The FF alpha spread between these portfolios is −2.25% per month, even after controlling for IV. The UMC portfolio in Panel B, constructed with reversed roles for expected entropy and IV, shows significant alphas at the 5% level. This indicates that expected IE more effectively explains the negative relationship between IV and returns than IV does in explaining the negative association between expected entropy and returns.

7.3.3 IE and lottery-like features. This paper uses two scenarios for analyzing the lottery-like feature of entropy that helps explain the IV puzzle. To conduct the analyses, we used the analyses of Bali *et al.* (2011). The first analysis is based on Panel A in Table 3, where Portfolio 1 (low IE) is the portfolio of stocks with the lowest IE during the past 60-day formation period, and portfolio 5 (high IE) is the portfolio of stocks with the highest IE during the past 60-day formation period. The raw return difference between decile 5 (high IE) and decile 1 (low IE) is −2.42% per month with a *t*-statistic of −4.49. In addition to the raw returns, Table 3 also presents the intercepts (Carhart four-factor alphas). As shown in the last row of Table 3, the difference in alphas between the high IE and low IE portfolios is −2.47% per month with a *t*-statistic of −4.46. This difference is economically significant and statistically significant at all conventional levels. Taking a closer look at the average returns and alphas across deciles, it is clear that the pattern is not one of a uniform declines as IE increases. The average returns of deciles 1–3 are approximately the same, in the range of 0.69–1.09% per month; but, going from decile 4 to decile 5, average returns drop significantly, from 0.69 to −1.08% and then to −1.33% per month. The alphas for the first three deciles are also almost similar, but again they fall dramatically for deciles 4 through 5. Given a preference for upside potential, investors may be willing to pay more for, and accept lower expected returns on, assets with these extremely high positive returns. In other words, it is conceivable that investors view these stocks as valuable lottery-like assets, with a small chance of a large gain.

The second alternative analysis of the extent to which a stock exhibits lottery-like payoffs is to compute IE over longer past periods. We construct the estimates $IE_{i,t+T}$ at the end of each month from January 1988 through June 2019, as outlined in (11) and (12), using 90-, 180-, and 365-day formation periods. The risk factors used in the cross-sectional regressions are those used in model 6 of Table 2. Next, we sort stocks into portfolios at the end of each month based on $E_t[IE_{i,t+T}]$ and calculate the value-weighted returns of each portfolio in month t+1. Panels A, B, and C of Table 12 report summary statistics for the five portfolios sorted into IE for 90-, 180-, and 365-day formation periods, respectively, where portfolio 1 represents stocks with the lowest predicted IE and portfolio 5 represents stocks with the highest predicted IE. Column 1 of each panel reports the time-series average of the value-weighted portfolio returns, and column 2 presents their corresponding idiosyncratic IE. Column 3 of each panel reports estimated alphas for the Carhart (1997) four-factor model. Although the economic

	Panel A: 90-day formation period			Panel B: 180-day formation period			Panel C: 365-day formation period		
	Mean	*IE*	Carhart alpha	Mean	*IE*	Carhart alpha	Mean	*IE*	Carhart alpha
1 (Low)	1.06	0.05	1.10 (3.53)*	1.00	0.06	1.04 (3.34)*	0.89	0.08	0.93 (3.00)*
2	0.86	0.15	0.91 (2.17)**	0.92	0.16	0.81 (2.01)**	0.77	0.18	0.79 (1.92)***
3	0.60	0.18	0.68 (1.77)***	0.53	0.22	0.60 (1.68)***	0.43	0.26	0.49 (1.38)
4	−1.12	0.19	−1.19 (−3.31)*	−1.02	0.24	−1.10 (−3.20)*	−1.00	0.23	−1.03 (−3.07)*
5 (High)	−1.24	0.33	−1.26 (−3.57)*	−1.12	0.30	−1.15 (−3.66)*	−1.10	0.34	−1.12 (−3.71)*
5–1	−2.20 (−4.02)*		−2.36 (−4.30)*	−2.12 (−3.55)*		−2.19 (−3.68)*	−1.99 (−2.22)**		−2.05 (−2.29)*

Note(s): We construct the estimates $IE_{i,t+T}$ at the end of each month from January 1988 through June 2019, as outlined in (11) and (12), using a 90-, 180, and 365-day formation period. The risk factors used in the cross-sectional regressions are those used in model 6 of Table 2. Next, we sort stocks into portfolios at the end of each month based on $E_t[IE_{i,t+T}]$ and calculate the value-weighted returns of each portfolio in month $t+1$. Panels A, B, and C of this table report summary statistics for the five portfolios sorted into IE, where portfolio 1 represents stocks with the lowest predicted IE and portfolio 5 represents stocks with the highest predicted IE. Column 1 of each panel reports the time-series average of the value-weighted portfolio returns, and column 2 presents their corresponding idiosyncratic IE. Column 3 of each panel reports estimated alphas for the Carhart (1997) four-factor model. *, **, and *** denote the significance of t-statistics (in parentheses) at the 1%, 5%, and 10% levels
Source(s): Created by the author

Table 12.
Descriptive statistics of portfolios were sorted into idiosyncratic risk measures

significance of these return differences reduces when we move from the 90-day formation period to 365-day formation period, we still see economic and statistical significance in IE. We can see that the average raw return differences are −2.20%, −2.12%, and −1.99% per month, respectively), and they are all statistically significant. More importantly, the differences between the four-factor Carhart alphas for the low and high IE portfolios are negative and economically and statistically significant for all formation periods. Specifically, the alpha differences for the IE portfolios are in the range of −2.05% to −2.36% per month, with t-statistics ranging from −2.29 to −4.30.

These analyses show that different proxies for lottery-like payoffs generate similar results, confirming their robustness and thus providing further support for the explanation we offer.

7.3.4 Discussion. The concept of “entropy,” originating from classical mechanics and information theory, is a measure of disorder or randomness within a system. For example, a stock market with predictable behavior and no irregularities would have a low IE, indicating a structured environment. On the other hand, a market with erratic stock prices demonstrates high IE, reflecting its unpredictability. Unlike general finance risk factors like volatility or market beta, which track overall asset movements, IE specifically zeroes in on the surprise element or unpredictability in stock movements. For instance, two stocks with similar volatility levels may differ in IE if one shows more erratic behavior. This is evident in real-world situations, such as a company’s stock price movement around earnings announcements or during mergers, or its debt repayment predictability, all of which can indicate varying levels of IE.

Table 11’s findings highlight IE’s significant role in understanding the lower average returns of stocks with high IV. The results also show that IV helps to explain the observed negative correlation between expected IE and returns. A notable point is that a risk factor can retain its explanatory power even when another factor is considered. Both factors – entropy and volatility – have overlapping estimates in their explanatory powers, making it challenging to attribute these to one specific factor. Since there's no established framework linking IE to expected returns, this paper contributes significantly to this area.

In exploring the IV puzzle, the entropy-based explanation adds to other theories addressing market imperfections, like short-selling constraints or varied investor opinions. Entropy preference is a key element in understanding why investors value stocks with high idiosyncratic risk. High valuations from entropy-preferring, under-diversified investors are more likely to be stable in high short-selling cost scenarios. Additionally, diverse investor opinions might indicate a wider range of entropy preferences, where under-diversified investors value potential stock gains higher than their well-diversified counterparts.

8. Conclusion

The risk management field has extensively studied IV and IS, but the empirical analysis exploring the connection between IE and stock returns has been largely overlooked. Our research aims to bridge this gap by implementing a model that uses predicted entropy to elucidate the cross-section of stock returns. Our findings reveal that lagged IV is a stronger predictor of entropy compared to lagged IE. Therefore, we incorporate IV and other risk factors as controls to predict IE. This approach to predicting entropy demonstrates significant pricing impacts, particularly in line with the Carhart four-factor model. When sorting stocks based on predicted entropy, we observe that the Carhart alpha for portfolios with lower entropy surpasses that of higher entropy portfolios by −2.47% per month. We identified a negative correlation between IE and expected returns, indicating that investors can potentially earn a premium from stocks with higher entropy levels. Delving deeper, our analysis suggests that this premium may be attributable to the fact that high IV is a reliable indicator of stocks likely to exhibit high future entropy exposure. Consequently, our results point to forecasted entropy as a key factor explaining the negative relationship between IV and expected returns. Although market imperfections might also influence this relationship, our findings hint that understanding investors' preferences could be a crucial starting point in unraveling this complex dynamic.

The discovery of a negative relationship between expected IE and stock returns carries profound implications for portfolio management and risk assessment strategies. Investors and financial analysts could leverage the insights derived from this study to refine their models for predicting stock performance. By incorporating measures of IE into these models, they can potentially achieve a more nuanced understanding of the risk-return trade-off associated with individual stocks. This approach could guide more informed decisions regarding asset allocation, particularly in the context of diversification strategies aimed at mitigating unsystematic risk. Furthermore, financial institutions might consider developing new financial products or investment tools that explicitly account for idiosyncratic entropy, thus offering investors more sophisticated means to manage their portfolios in alignment with their risk preferences.

The notable connection between exposure to IE and expected returns presents a compelling topic for future empirical and theoretical research. Such studies should aim to thoroughly explore the underlying factors driving this risk and how they relate to stock risk premiums. Consequently, developing co-entropy risk measures and investigating their correlations with stock risk premiums would be a valuable direction for upcoming research endeavors.

Notes

1. The IV puzzle has attracted considerable attention among researchers, a fact highlighted by notable studies such as those conducted by Lee and Mauck (2016) and Aabo, Pantzalis, and Park (2017).
2. To prove this rule, see Taylor (1952).
3. Shannon and Weaver (1964) have clarified why the logarithmic form is used in entropy calculations. It's important to note that since the logarithm of a fraction is negative and the probability $p_{i,d,t}$ is less than 1, the value of $IE_{i,t}(X^*)$ will always be greater than 0. For example, if a system consistently produces only one specific event, there will be no uncertainty, and therefore $IE_{i,t}(X^*)$ equals 0. In line with this concept, entropy increases by one unit whenever the count of identical events doubles. Ultimately, the entropy measure reaches its highest value when all possible events have an equal chance of occurring.
4. Panel D in Figure 1 also shows similar results, indicating that episodes of high IV and high IV dispersion align with the findings of Brandt, Brav, Graham, and Kumar (2008). They describe these times as "speculative periods."
5. Maximum (MAX) is the maximum daily return within a month by using $MAX_{it} = \max(R_{id}), d = 1, \ldots, D_t$ and Minimum (MIN) is the negative of the minimum daily return within a month by using $MIN_{it} = -\min(R_{id}), d = 1, \ldots, D_t$, where R_{id} is the return on stock i on day d and D_t is the number of trading days in month t.
6. We carried out thorough checks to verify the appropriateness of our chosen formation period and found consistent results across various periods, including 30, 90, 150, and 365 days for IE formation. These findings are detailed in subsection (4.3.1).
7. The results of these checks are not reported in this paper and are available on request.
8. The results are not reported in this paper and are available on request.
9. The results are not reported due to space restrictions and are available on request.
10. While various values of α are employed in academic studies, both Principe (2010) and Chernyshov (2012) demonstrate that setting α to 2 leads to favorable outcomes in empirical research.
11. Return entropy comprises both IE and additional co-entropy elements. While Table 2 indicates that IV is a predictor of IE, further investigation is required to ascertain whether IV can also forecast co-entropy returns. This aspect is earmarked for exploration in future research.

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Corresponding author
Mohammadreza Tavakoli Baghdadabad can be contacted at: m.tavakoli.b@westernsydney.edu.au