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American put options with regime-switching volatility

Bong-Gyu Jang and Hyeng Keun Koo
Department of Industrial and Management Engineering,
Pohang University of Science and Technology, Pohang, Republic of Korea

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Abstract

We present an approach for pricing American put options with a regime-switching volatility. Our method reveals that the option price can be expressed as the sum of two components: the price of a European put option and the premium associated with the early exercise privilege. Our analysis demonstrates that, under these conditions, the perpetual put option consistently commands a higher price during periods of high volatility compared to those of low volatility. Moreover, we establish that the optimal exercise boundary is lower in high-volatility regimes than in low-volatility regimes. Additionally, we develop an analytical framework to describe American puts with an Erlang-distributed random-time horizon, which allows us to propose a numerical technique for approximating the value of American puts with finite expiry. We also show that a combined approach involving randomization and Richardson extrapolation can be a robust numerical algorithm for estimating American put prices with finite expiry.

Keywords Derivative pricing, American option, Regime switch, Stochastic volatility

Paper type Research paper

1. Introduction

The coming of modern option pricing theory, pioneered by Black and Scholes (1973) and Merton (1973), has had a profound impact on both the academic discipline of finance and the practical operations of the financial services industry. The foundation of this theory, rooted in the concept of the absence of arbitrage opportunities, has evolved into a dominant paradigm in the field of finance, as exemplified by seminal works such as Harrison and Kreps (1979) and Dybvig and Ross (2003). This transformative technology for pricing and risk management has empowered financial institutions to develop a diverse array of customized contracts and securities, tailored precisely to meet the unique requirements of their clientele. In this light, option pricing stands as a fundamental cornerstone of modern finance.

In the early stages of option pricing analysis, exemplified by the renowned Black–Scholes formula, a fundamental assumption prevailed: that the volatility of the underlying asset remained constant. Nevertheless, the veracity of this assumption has been cast into doubt, primarily due to compelling empirical evidence revealing that implied volatility, as observed across various strike prices and maturities, does not exhibit constancy.

In response to the challenges posed by this empirical evidence, extensive research endeavors have been undertaken to expand the scope of the Black–Scholes–Merton framework to encompass scenarios featuring randomly fluctuating volatility. Substantial progress has been made in this pursuit, with notable contributions from scholars such as Hull

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and White (1987), Wiggins (1987), Stein and Stein (1991), Heston (1993), Bates (1996) and Duffie *et al.* (2000). These efforts have significantly advanced our understanding and modeling capabilities in the context of dynamic and variable volatility in financial markets.

We contribute to the field of option pricing by conducting an investigation into the pricing of American options in the context of stochastic volatility. Our specific focus on stochastic volatility entails a unique characteristic: the volatility of the underlying asset exhibits a binary behavior. It transitions between two distinct states, denoted as the "high volatility regime" (H) and the "low volatility regime" (L), at the occurrence of jumps dictated by an independent Poisson process. The regime-switching volatility model can be considered as the simplest version of the stochastic volatility model. This model is useful when the underlying asset undergoes regime shifts over a period or when the maturity of the underlying asset is long enough so that the option lifetime includes abrupt environmental changes. Compared to parameter-rich volatility models like the Heston (1993) model, it has relatively fewer parameters, making it considerably easier to calibrate from options market data.

However, it is important to note that our model does not encompass the full spectrum of randomness associated with volatility. For instance, it does not account for potential correlations between changes in volatility and alterations in the underlying asset's price. Nevertheless, our exploration of American option pricing within the context of regime-switching volatility represents an initial step toward comprehending the valuation of American options in the presence of stochastic volatility. This research sets the foundation for further, more intricate models that can capture additional nuances in the relationship between option pricing and the dynamic nature of volatility.

Option pricing in the context of regime switching has attracted the attention of several researchers, including Naik (1993), Chourdakis and Tzavalis (2000), Chourdakis (2001), Campbell and Li (2002) and Edwards (2005). Empirical findings from their work have shown promise, indicating that regime-switching models offer a fitting representation of market data comparable to more intricate models like stochastic volatility models with jumps. The papers, however, has primarily concentrated on European options.

In a distinctive departure, Bollen (1998) introduced a novel approach known as the "pentanomial tree" to calculate the price of American options within a regime-switching framework. While Bollen presented numerical results, analytical solutions were not derived. Building upon this foundation, Bollen *et al.* (2000) extended the application of Bollen's pentanomial trees to investigate currency option prices. Duan *et al.* (2002) have contributed to this field by proposing a numerical methodology for pricing American options in a discrete-time regime-switching volatility model. Similarly, Driffill *et al.* (2002) have introduced a closed-form solution for perpetual American call options in scenarios where the dividend process follows a regime-switching pattern. Lastly, Guo and Zhang (2004) have made significant strides by deriving analytic formulas for perpetual American put options within the context of regime-switching volatility. These collective efforts have enriched our understanding of option pricing under regime switching and have paved the way for further exploration in this intriguing area of finance.

We establish fundamental theoretical insights within the framework of regime-switching volatility. Specifically, we demonstrate that the price of an American put option can be represented as the combination of a European put option price and the premium attributed to the early exercise privilege. This characterization extends findings established by Kim (1990), Jacka (1991) and Carr *et al.* (1992) within the context of constant volatility. Moreover, we provide a rigorous proof that the valuation of an American put option constitutes a unique solution to the free boundary value problem. This result builds upon earlier work by Mckean (1965) and Van Moerbeke (1976), expanding its applicability to the regime-switching volatility setting. Furthermore, we delve into the realm of perpetual American put options, demonstrating that their pricing exhibits distinct characteristics across the volatility regimes. Specifically, we establish that perpetual put options command a higher price during periods of high volatility compared to low volatility phases. Additionally, we reveal that the optimal exercise boundary is situated at a lower threshold in high-volatility regimes, as opposed to low-volatility regimes. These findings contribute valuable insights into the dynamics of option pricing in the context of fluctuating volatility regimes (Theorems 3.7 and 3.8).

To facilitate the numerical determination of the American put option's value, we employ the randomization method introduced by Carr (1998). This method involves substituting the fixed time horizon with a random time, effectively transforming the problem into one that resembles the pricing of a perpetual put option. Consequently, the pricing of an American put option with a specified time horizon, denoted as T, can be approximated through a backward induction process. This process considers a substantial number, N, of randomly generated time points, whose summation is expected to equal the original time horizon T, with a variance equal to T/N. We derive an analytical expression for the value of an American put option with such a random time horizon. Subsequently, we present numerical examples that showcase the robustness of the randomization method in conjunction with Richardson extrapolation. This combined approach serves as a robust algorithm for approximating the price of American put options with finite maturities in the regime-switching model. Our analysis includes a comparative evaluation of the numerical results obtained through this method with those generated using the pentanomial tree proposed by Bollen (1998). Furthermore, we establish a noteworthy observation: the N-th approximation of an American put option with finite maturity consistently commands a higher price during periods of high-volatility compared to low-volatility phases. Additionally, we demonstrate that the optimal exercise boundary is situated at a lower threshold in high-volatility regimes relative to low-volatility regimes (Theorem 4.4) [1].

A paper closely related to ours is Boyarchenko and Levendorskii (2009), which presents an American option pricing framework within the context of Markov-modulated Lévy models, utilizing Carr's randomization procedure. While our paper shares certain similarities with their work, it distinguishes itself by offering a unique contribution—analytical comparisons of the optional exercise boundaries and option prices, contingent on the volatility regime.

After Boyarchenko and Levendorskii (2009), Huang *et al.* (2011) introduce iterative procedures to value American options under regime switching. Zhang *et al.* (2014) exploit a penalty method to solve a system of complementarity problems arising from pricing American options. Yousuf *et al.* (2015) develop a second-order method based on an exponential time-differencing approach for solving American options under multi-state regime switching. Chiarella *et al.* (2016) solve the American option pricing problem under regime-switching by using the method-of-lines scheme. Lu and Putri (2020) use the Laplace transform method to solve the system of the partial differential equations for American option pricing. Including these papers, there are numerous papers dealing with the valuation of American options under regime switching. However, most of them do not show analytical comparisons based on the regimes covered in this paper.

The paper is organized as follows. Section 2 explains the model with regime-switching volatility. Section 3 studies a perpetual American put. Section 4 investigates pricing of an American put with finite expiry by randomization of time horizon and Section 5 discusses its implementation and shows numerical examples. Finally, Section 6 concludes. All the proofs are contained in the Appendix.

2. Model

In this section, we explain a model of a financial market. In the model, all activity occurs on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}$ is the augmentation of the filtration generated by a (one-dimensional) Brownian motion B and two Poisson processes $\varphi_H(\cdot)$ and

$\varphi_L(\cdot)$ on $[0, \infty)$, where the two Poisson processes are independent of each other and independent of the Brownian motion B.

In the financial market, there are two (underlying) assets. The first asset (the bond) is a money market account and is instantly risk-free. We assume that the return on the riskless asset is constant and equal to r. The second asset (the stock) is risky and American options are written on this asset. The price S_t of the stock evolves according to the following equation

$$\frac{dS(t)}{S(t)} = \mu(t)dt + \sigma(t)dB(t), \quad S(0) = S_0,$$

where $\mu(t)$, $\sigma(t)$ are $\{\mathcal{F}_t\}$-predictable processes. The stock pays no cash dividends [2].

The volatility of the second asset takes two values $\sigma_H > \sigma_L > 0$; namely, there are two regimes, say, "high volatility regime" H and "low volatility regime" L and volatility σ_H in regime H is larger than volatility σ_L in regime L, i.e. $\sigma_H > \sigma_L > 0$. This assumption implies, in particular, $\sigma(0) = \sigma_0 = \sigma_{i_0}$ for some $i_0 \in \{H, L\}$, that is, the initial volatility regime is i at time 0. The volatility changes according to a Markov regime-switching process; regime i switches to regime $j(\neq i)$ at the next jump time of Poisson process φ_i for $i, j \in \{H, L\}$, i.e. suppose at time t, the market is in regime i and $\varphi_i(t) = n$ $(n = 0, 1, 2, \cdots)$ then the volatility regime changes to $j \neq i$ at time $\inf\{s\colon s \geq t\ \varphi_i(s) = n + 1\}$. Under P, the intensity of the Poisson process φ_i is assumed to be $\tilde{\lambda}_i(t)$ at time t.

We assume that there is no market frictions, i.e. no taxes, no transaction costs and no short-sale constraints. Portfolio strategies, consumption processes and self-financing strategies with an initial wealth are defined as in a standard financial model (see, e.g. Chapter 1, Karatzas and Shreve, 1998).

We employ a standard assumption in modern finance, that is, there is no arbitrage opportunity in the financial market. Then, under a suitable regularity condition, there exists a probability measure Q such that for any financial contract $\tilde{S}(t)$ is a Q-martingale, where $\tilde{S}(t)$ is the time-t market price of the financial contract (see, e.g. Harrison and Pliska, 1981) [3]. The martingale measure may not be unique if the market is incomplete (see Chapter 5, Karatzas and Shreve, 1998). We will not assume the completeness of the financial market in this paper. However, we will assume the existence of a unique martingale measure that is used for pricing of financial contracts by market participants.

We will assume that the following is valid throughout the paper:

Standing assumption. There is a unique probability measure under which all the price of every financial contract is a martingale. The measure is used for pricing of all securities that will be introduced to the market. Under Q, the intensity of the Poisson process φ_i is λ_i for $i \in \{H, L\}$.

We will call Q the *risk-neutral* probability measure and the stock price process $S(t)$ satisfies

$$\frac{dS(t)}{S(t)} = rdt + \sigma_i d\tilde{B}(t), \quad \text{in regime} \quad i,$$

where $\tilde{B}$ is a standard Brownian motion under Q. Note that a jump occurs only to the volatility of the stock, not to its price.

We now consider an American put option with maturity at $T < \infty$ and strike price K written on the stock. Suppose that the stock price is equal to S and the volatility regime is $i \in \{H, L\}$ at time 0, then we define

$$P_i(T-t, S) \equiv \operatorname*{ess\,sup}_{t \leq \tau \leq T} E^Q\left[e^{-r(\tau-t)}(K - S_\tau)^+ \,\middle|\, \mathcal{F}_t\right](\omega), \quad \text{for} \quad \omega \in \{S(t) = S\} \cap \{\sigma(t) = \sigma_i\},$$

where τ is an $\{\mathcal{F}_s\}_{t\leq s\leq T}$-stopping time. By the Markov property of the Brownian motion and Poisson processes, $P_i(T-t, S)$ is uniquely determined. $P_i(T-t, S)$ will be called the *fair price* of the American put at time t when the stock price and volatility regime at t are S and i, respectively.

The next proposition shows that our definition of the fair price is equal to the smallest wealth to hedge a short position of the American put if the market is *complete*. In order to state the proposition, we will consider (possibly non-existent) financial assets Θ_i for $i = H$ or L whose price is perfectly correlated with the Poisson process φ_i. The financial market is complete if both Θ_H and Θ_L are *tradable*, i.e. the price processes exist and investors can form their portfolios using the assets.

Proposition 2.1. *Suppose that the financial market is complete, i.e. for $j = H$ and L* the financial contract Θ_j is tradable in the market. Suppose also that $S_0 = S$, $\sigma(0) = \sigma_i (i \in \{H, L\})$ Then, $P_i(t, S)$ is equal to the smallest initial wealth required such that there exists a self-financing strategy whose wealth process $X(u)$ satisfies

(i). $X(u) \geq (K - S(u))^+$ *for all* $0 \leq u < T$.

(ii). $X(T) = (K - S(T))^+$.

Now we state a lemma that will be useful to define the optimal exercise boundaries.

Lemma 2.2. For $i \in \{H, L\}$, $P_i(T, S)$ is Lipschitz continuous in S, uniformly continuous in T, nondecreasing in T, nonincreasing in S and convex in S.

Let

$$\mathcal{C}_i \equiv \left\{(u, S) \in [0, \infty)^2 : P_i(u, S) > (K - S)^+\right\}$$
$$\mathcal{S}_i \equiv \left\{(u, S) \in [0, \infty)^2 \times [0, T) : P_i(u, S) = (K - S)^+\right\},$$

for $i \in \{H, L\}$. By Lemma 2.2 $\mathcal{S}_i$ is closed and $\mathcal{C}_i$ is open and $\mathcal{S}_{i,u} \equiv \{S : (u, S) \in \mathcal{S}_i\}$ and $\mathcal{C}_{i,u} \equiv \{S : (u, S) \in \mathcal{C}_i\}$ are connected for every $u \in [0, \infty)$

For $i \in \{H, L\}$, let us define the *optimal exercise boundary* $\underline{S}_i$: for $u \in [0, \infty)$

$$\underline{S}_i(u) \equiv \sup\{S : S \in \mathcal{S}_{i,u}\}.$$

Then, the *optimal stopping time* τ^* for exercising the American put can be characterized as follows:

$$\tau^* = \min_{i\in\{H,L\}} \left\{ \inf_{0\leq u\leq T} \left\{ u : \sigma(u) = \sigma_i \ \ \& \ \ S_u \leq \underline{S}_i(u) \right\} \right\}.$$

The following theorem states that the American put option price is the sum of the price of a European put option and the premium for the early exercise privilege. The results have been obtained by Kim (1990), Jacka (1991) and Carr *et al.* (1992) in the constant volatility case. For a measurable set, $A \in \mathcal{F}_t$, let χ_A denotes its characteristic function, i.e. $\chi_A(x) = 1$, if $x \in A$ and $\chi_A(x) = 0$ if $x \in \Omega - A$

Theorem 2.3. For $i \in \{H, L\}$, let $p_i(T, S)$ be the fair price of a European option with a strike price equal to K, time to maturity equal to T and the underlying state S at time T, when the current regime is i. Then,

$$P_i(T,S) = p_i(T,S) + \int_0^T e^{-rt} rK\chi_{\left\{S(t)<\underline{S}_H \& \sigma(t)=\sigma_H\right\} \cup \left\{S(t)<\underline{S}_L \& \sigma(t)=\sigma_L\right\}} dt$$

In the constant volatility case, Mckean (1965) and Van Moerbeke (1976) have shown that the fair value of the American put is a solution to a *free boundary value problem*. We establish a similar result for the case with regime-switching volatility in the following. Let us define a differential operator $\mathcal{L}$ for a pair $(f_H(u,S), f_L(u,S))$ of functions which are C^2 on a subset of $\mathbb{R}^2$: for $i \in \{H, L\}$

$$\mathcal{L}f_i \equiv \frac{\sigma_i^2}{2} S^2 f_{i,SS} + rSf_{i,S} - rf_i + \lambda_i\left(f_j - f_i\right) - f_{i,u},$$

where $j \in \{H, L\}, j \neq i$. Here, subscripts denote partial derivatives, i.e. $f_{i,SS} = \frac{\partial^2 f_i}{\partial S^2},\ f_{i,S} = \frac{\partial f_i}{\partial S}$ and $f_{i,u} = \frac{\partial f_i}{\partial u}$.

Then, we consider the following free boundary value problem.

Problem 2.4. Find a pair $(f_H(u,S), f_L(u,S))$ of continuous functions defined on $[0,\infty)^2$ and a pair $(\mathcal{D}_H(u), \mathcal{D}_L(u))$ of boundaries defined on [0, T] such that

$$\mathcal{L}f_i(u,S) = 0 \quad \text{for} \quad S > \mathcal{D}_i(u), \quad i \in \{H, L\}$$

subject to the following six boundary conditions

$$\lim_{S\uparrow\infty} \max_{0\le u\le T} |f_i(u,S)| = 0,\ \forall T < \infty \quad \lim_{S\downarrow\mathcal{D}_i(u)} f_i(u,S) = K - \mathcal{D}_i(u), \quad \lim_{S\downarrow\mathcal{D}_i(u)} f_{i,S}(u,S) = -1,$$

and four terminal conditions

$$\mathcal{D}_i(T) = K, \quad f_i(T,S) = (K-S)^+, \quad \text{for} \quad S > 0.$$

Furthermore,

$$f_i(u,S) = K - S, \quad \text{for} \quad S \le \mathcal{D}_i(u),\ u \in [0,T],$$

and

$$f_i(u,S) \ge (K-S)^+, \quad \text{for} \quad u \in [0,T],\ S > 0.$$

Proposition 2.5. $(P_H(u,S), P_L(u,S))$ and $\left(\underline{S}_H(u), \underline{S}_L(u)\right)$ is the unique solution to Problem 2.4.

3. Perpetual American puts

In this section, we consider the case where the option's lifetime is infinite, that is, $T = \infty$. McKean (1965) and Merton (1973) derived an analytic form for the option price for the case where the volatility is constant. In this section, we derive an analytic form for the value of a perpetual American put as in Guo and Zhang (2004). Furthermore, we provide a result concerning the existence and uniqueness of a solution to the free boundary value problem (Theorem 3.5) and show that the price of the perpetual put is always higher in the high-volatility regime than in the low-volatility regime and the exercise boundary is lower in the high-volatility regime than in the low-volatility regime (Theorems 3.7 and 3.8).

We will use the notation $P_i(S)$ for the value of American put and $\underline{S}_i$ for the free boundary in regime $i \in \{H, L\}$, omitting time variable t. The notation is justified because the value and optimal exercise boundary of the perpetual American put option are time-homogeneous, i.e. independent of time t, given the volatility regime and stock price.

Now we consider the following free boundary value problem that will characterize the value of a perpetual American put option.

Problem 3.1. Find a pair ($f_H(S)$, $f_L(S)$) of continuous functions defined on $(0, \infty)$ and a pair $(\mathcal{D}_H, \mathcal{D}_L)$ of positive real numbers such that

$$\mathcal{L}^P f_i(S) = 0 \quad \text{for} \quad S > \mathcal{D}_i, \quad i \in \{H, L\} \tag{3.1}$$

where

$$\mathcal{L}^P f_i \equiv \frac{\sigma_i^2}{2} S^2 f_{i,SS} + rSf_{i,S} - rf_i + \lambda_i (f_j - f_i)$$

with $j \neq i$, subject to the following six boundary conditions

$$\lim_{S\uparrow\infty} |f_i(S)| = 0, \quad \lim_{S\downarrow\mathcal{D}_i} f_i(S) = K - \mathcal{D}_i, \quad \lim_{S\downarrow\mathcal{D}_i} f_{i,S}(S) = -1, \tag{3.2}$$

Furthermore,
$$f_i(S) = K - S, \quad \text{for} \quad S \leq \mathcal{D}_i,$$

and

$$f_i(S) \geq (K - S)^+, \quad \text{for} \quad S > 0.$$

Proposition 3.2. ($P_H(S)$, $P_L(S)$) and $\left(\underline{S}_H, \underline{S}_L\right)$ is the unique solution to Problem 3.1.

We will later show that $\underline{S}_L > \underline{S}_H$. However, we still do not know which of $\underline{S}_L$ and $\underline{S}_H$ is larger and will temporarily let $\underline{S}_M \equiv \max\left\{\underline{S}_H, \ \underline{S}_L\right\}$ and $\underline{S}_m \equiv \min\left\{\underline{S}_H, \ \underline{S}_L\right\}$. Namely, M and m will denote H or L such that $\underline{S}_M \geq \underline{S}_m$ (When $\underline{S}_M = \underline{S}_m$, we will choose notation such that $M = L$ and $m = H$).

In the following, we will derive the unique solution ($P_H(S)$, $P_L(S)$) and $\left(\underline{S}_H, \underline{S}_L\right)$ to Problem 3.1 in explicit form. An alternative derivation can be found in Guo and Zhang (2004).

Lemma 3.3. There exist two real negative numbers n_1 and n_2($n_1 > n_2$) such that a general solution to equations (3.1) for $S > \underline{S}_M$ subject to boundary conditions $\lim_{S\uparrow\infty} P_i(S) = 0$ for $i, j \in \{H, L\}$ are given by

$$P_H(S) = C_1 S^{n_1} + C_2 S^{n_2}, \quad P_L(S) = C_1 \xi_1 S^{n_1} + C_2 \xi_2 S^{n_2}. \tag{3.3}$$

for some $C_1, C_2 \in \mathbb{R}$. Furthermore, n_1, n_2, ξ_1 and ξ_2 satisfy

$$\begin{gathered} n_2 < -2r/\sigma_L^2 < n_1 < -2r/\sigma_H^2, \\ 1 > \xi_1 > 0 > \xi_2. \end{gathered} \tag{3.4}$$

Lemma 3.4. Suppose that $\underline{S}_m < S < \underline{S}_M$. Then, the value of the perpetual American put $P_m(S)$ in regime m takes the following form:

$$P_m(S) = D_1 S^{m_1} + D_2 S^{m_2} - S + \frac{K\lambda_m}{r+\lambda_m} \tag{3.5}$$

for some constants D_1 *and* D_2, *where*

$$m_1 = \frac{\sigma_m^2 - 2r + \sqrt{a_m}}{2\sigma_m^2}, \quad m_2 = \frac{\sigma_m^2 - 2r - \sqrt{a_m}}{2\sigma_m^2}, \quad a_m = \left(2r - \sigma_m^2\right)^2 + 8\sigma_m^2(r+\lambda_m) > 0.$$

Furthermore, m_1, m_2 *satisfy*

$$m_1 > 1, \;\; m_2 < 0. \tag{3.6}$$

Now we proceed to determine constants $\underline{S}_m$, $\underline{S}_M$, C_1, C_2, D_1, D_2. Let us define

$$f(z,s) \equiv \frac{r+\lambda_m}{rzK}\left(\frac{(z-n_1)(n_2(K-s)+s)}{(n_2-n_1)\xi_1} + \frac{(z-n_2)(n_1(K-s)+s)}{(n_1-n_2)\xi_2} + (z-1)s - \frac{z\lambda_m K}{r+\lambda_m}\right).$$

The following result shows how to determine the solution to the free boundary problem.

Proposition 3.5. $\underline{S}_M$, $\underline{S}_m$, C_1, C_2 satisfy the followings:

$$\left[f\left(m_1, \underline{S}_M\right)\right]^{m_1} = \left[f\left(m_2, \underline{S}_M\right)\right]^{m_2},$$

$$\left(\frac{\underline{S}_M}{\underline{S}_m}\right)^{m_2} = f\left(m_1, \underline{S}_M\right), \qquad \left(\frac{\underline{S}_M}{\underline{S}_m}\right)^{m_1} = f\left(m_2, \underline{S}_M\right), \tag{3.7}$$

and

$$C_1 = \frac{n_2\left(K - \underline{S}_M\right) + \underline{S}_M}{(n_2 - n_1)\xi_1 \underline{S}_M^{n_1}}, \quad C_2 = \frac{n_1\left(K - \underline{S}_M\right) + \underline{S}_M}{(n_1 - n_2)\xi_2 \underline{S}_M^{n_2}}, \tag{3.8}$$

$$D_1 = \frac{r m_2 K}{(r+\lambda_m)(m_2 - m_1)\underline{S}_m^{m_1}}, \quad D_2 = \frac{r m_1 K}{(r+\lambda_m)(m_1 - m_2)\underline{S}_m^{m_2}}. \tag{3.9}$$

Corollary 3.6. (i) Both D_1 and D_2 are positive and (ii) $C_1 \neq 0$ and $C_2 \neq 0$.

We now show in the following theorems that the price of the perpetual put is always higher in the high-volatility regime than in the low-volatility regime and the optimal exercise boundary is lower in the high-volatility regime than in the low-volatility regime.

Theorem 3.7. $\underline{S}_L > \underline{S}_H$, *i.e.* $m = H$ *and* $M = L$.

Theorem 3.8. $P_H(S) \geq P_L(S)$ for all $S > 0$ and the inequality is strict for $S > \underline{S}_H$.

In Figure 1, we draw a graph of the value of a perpetual American put option where the parameters are given as $r = 0.1, K = 1, T = 1, \sigma_H = 0.4, \sigma_L = 0.2, \lambda_H = 1.0$ and $\lambda_L = 0.5$. The intensity parameters in the figure are chosen such as the average time it takes a regime to

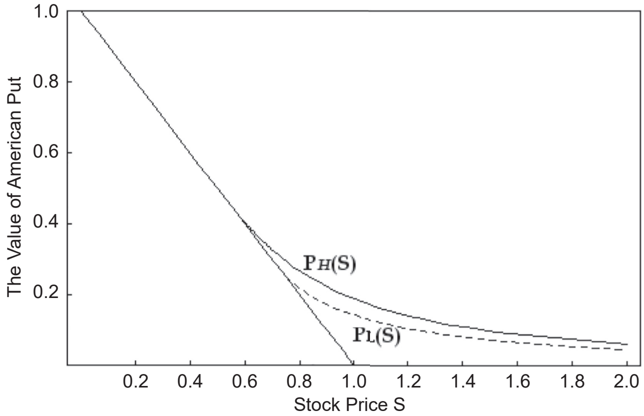


Figure 1.
Value of a perpetual American put option

Note(s): The parameters are r = 0.1, K = 1, T = 1, σH = 0.4, σL = 0.2, λH = 1.0 and λL = 0.5
Source(s): Authors' own work

change from high volatility to low volatility is a year, while the average time it takes a regime to change from low volatility to high volatility is half a year. The higher intensity λ_H in the high volatility regime H is supported by empirical studies, for example, Ang and Bekaert (2002). The result shown in the figure is consistent with Theorems 3.7 and 3.8.

These results are also consistent with our intuition: High volatility yields a higher time value, so the price P_H is higher and the corresponding optimal exercise boundary is lower. In the continuation region (the upper region of the optimal exercise boundary), a lower optimal exercise boundary gives the put holder a higher time value because the current stock price is above it.

4. American puts with finite expiry

In this section, we will study the valuation of an American put which has a finite expiration time T. In the case where volatility is constant Carr (1998) has studied the same problem by considering an American put with a random expiration time which is the sum of N independent random times that are identically distributed according to an exponential distribution with a mean equal to T/N. The random time is distributed according to an Erlang distribution with a mean equal to T and a variance equal to T/N, and therefore, the random-time approaches to T in probability as $N \to \infty$. He has suggested that the value of an American put with finite time T to expiration can be approximated by the value of an American put with the previously explained random time if N is sufficiently large. We will apply his technique to study the value of an American put with finite expiry.

4.1 Exponentially distributed time

In this section, we will consider an American put with a random time τ until expiration, where

$$P[\tau \in dt] = \beta e^{-\beta t} dt$$

with $\beta \equiv 1/T$. Namely, there is an independent Poisson process with an intensity β such that the option expires at the first jump time of the process [4].

Note that the expected time to expiration of the American put is β. Analysis of the American put with a random expiration time is very similar to that of the perpetual option in the previous section. As in the case of the perpetual option, the value and early exercise boundary of the American put option are time-homogeneous, i.e. independent of time t, given the volatility regime and stock price. Therefore, the equations describing the fair value of the American put are ordinary differential equations without having time derivatives. We will use notation $P_i^1(S)$ and $\underline{S}_i^1$ to denote the value and early exercise boundary of the American put, respectively, in regime $i \in \{H, L\}$.

As in Carr (1998), the free boundary problem for the fair price of the American put takes the following form: for $i, j \in \{H, L\}, i \neq j$,

$$\begin{cases} \frac{1}{2}\sigma_i^2 S^2 P_{i,SS}^1(S) + rSP_{i,S}^1(S) - (r+\lambda_i+\beta)P_i^1(S) + \lambda_i P_j^1(S) = 0 & \text{if } S > K \\ \frac{1}{2}\sigma_i^2 S^2 P_{i,SS}^1(S) + rSP_{i,S}^1(S) - (r+\lambda_i+\beta)P_i^1(S) + \lambda_i P_j^1(S) = -\beta(K-S) & \text{if } \underline{S}_i^1 < S < K \\ P_i^1(S) = K - S & \text{if } S < \underline{S}_i^1, \end{cases}$$

with boundary conditions

$$\lim_{S\uparrow\infty} P_i^1(S) = 0, \quad \lim_{S\downarrow \underline{S}_i^1} P_i^1(S) = K - \underline{S}_i^1, \quad \lim_{S\downarrow \underline{S}_i^1} P_{i,S}^1(S) = -1.$$

As in the previous section, let $\underline{S}_M^1 \equiv \max\left\{\underline{S}_H^1, \ \underline{S}_L^1\right\}$ and $\underline{S}_m^1 \equiv \min\left\{\underline{S}_H^1, \ \underline{S}_L^1\right\}$.

We will make the following assumption in this section.

Assumption I. There exist four distinct real roots n_1, n_2, n_3, n_4 of the algebraic equation $I_F(n) = 0$, where

$$I_F(n) = \left(\frac{1}{2}\sigma_H^2 n^2 + \left(r - \frac{1}{2}\sigma_H^2\right)n - (r + \lambda_H + \beta)\right)\left(\frac{1}{2}\sigma_L^2 n^2 + \left(r - \frac{1}{2}\sigma_L^2\right)n - (r + \lambda_L + \beta)\right) - \lambda_H \lambda_L,$$

and among them the two roots $n_1, n_2 (n_1 > n_2)$ are negative and others $n_3, n_4 (n_3 > n_4)$ are greater than 1.

Note the fact that $I_F(0) > 0$ and $I_F(1) > 0$. Thus, if the roots of equation $I_F(n) = 0$ exist in one of the open intervals $(-\infty, 0), (0, 1)$ and $(1, \infty)$, then there are exactly two or four. Assumption I tells us that we assume there exist exactly two roots in $(-\infty, 0)$ and the other two roots in $(1, \infty)$, respectively.

Proposition 4.1. Suppose that Assumption I is valid and let ξ_1, ξ_2, ξ_3 and ξ_4 be unique solutions to the following equation:

$$\begin{cases} \frac{1}{2}\sigma_H^2 n(n-1) + rn - r - \lambda_H - \beta + \lambda_H \xi = 0 \\ \xi\left(\frac{1}{2}\sigma_L^2 n(n-1) + rn - r - \lambda_L - \beta\right) + \lambda_L = 0. \end{cases}$$

Then the fair value of the American put with random expiration time τ takes the following form.

(i) In the region $S > K$, for some constants C_1^1, C_2^1

$$P_m^1(S) = C_1^1 S^{n_1} + C_2^1 S^{n_2}, \quad P_M^1(S) = C_1^1 \xi_1 S^{n_1} + C_2^1 \xi_2 S^{n_2}.$$

(ii) In the region $\underline{S}_M^1 < S < K$, for some constants $\tilde{C}_1^1,\ \tilde{C}_2^1,\ \tilde{C}_3^1$ and $\tilde{C}_4^1$,

$$\begin{cases} P_m^1(S) = \tilde{C}_1^1 S^{n_1} + \tilde{C}_2^1 S^{n_2} + \tilde{C}_3^1 S^{n_3} + \tilde{C}_4^1 S^{n_4} - S + \dfrac{\beta K}{r+\beta}, \\ P_M^1(S) = \tilde{C}_1^1 \xi_1 S^{n_1} + \tilde{C}_2^1 \xi_2 S^{n_2} + \tilde{C}_3^1 \xi_3 S^{n_3} + \tilde{C}_4^1 \xi_4 S^{n_4} - S + \dfrac{\beta K}{r+\beta}. \end{cases}$$

(iii) In the region $\underline{S}_m^1 < S < \underline{S}_M^1$, for some constants D_1^1 and D_2^1,

$$P_m^1(S) = D_1^1 S^{m_1} + D_2^1 S^{m_2} - S + \frac{K(\lambda_m + \beta)}{r + \lambda_m + \beta},$$

where $m_1 = \frac{\sigma_m^2 - 2r + \sqrt{a_m}}{2\sigma_m^2}$ and $m_2 = \frac{\sigma_m^2 - 2r - \sqrt{a_m}}{2\sigma_m^2}$, $a_m = (2r - \sigma_m^2)^2 + 8\sigma_m^2(r + \lambda_m + \beta) > 0$ and

$$P_M^1(S) = K - S.$$

There are 10 equations that can be derived from the smoothness of $P_M^1(\cdot)$ and $P_m^1(\cdot)$. And by some heavy calculation, we can reduce the 10 equations to two similar to the case of a perpetual put [5].

Theorem 4.2. Suppose that Assumption I is valid. Then, $\underline{S}_L^1 > \underline{S}_H^1$ and $P_H^1(S) \geq P_L^1(S)$ *for all* $S > 0$. The inequality is strict for $S > \underline{S}_H^1$.

4.2 Erlang-distributed time

In this section, we will consider an American put with a random expiration time τ which is the sum of N independent random-times which are identically and exponentially distributed with a mean equal to T/N. That is, the American put will expire after the N-th jump time of an independent Poisson process with intensity $\beta_N = N/T$. Then, τ is distributed according to an Erlang distribution, that is,

$$P[\tau \in dt] = \frac{(\beta_N)^N}{(N-1)!} t^{N-1} e^{-\beta_N t} dt.$$

Let us consider, N put options with the same strike price K. The first option is the put with an exponentially distributed time to expiration with $\beta = \beta_N$ explained in the previous section. The k-th option for $1 < k \leq N$ is a put which can be exercised for $(K - S)^+$ at any time up to and including the k-th jump time of a Poisson Process with intensity β_N, and if the option is not exercised after the k-th jump time of the Poisson Process it becomes the $(k-1)$-th put after the jump. Let $P_i^k(S)(k = 2, 3, \ldots, N)$ denote the fair value of the k-th American put in regime $i \in \{H, L\}$. We conjecture that the N-th put value $P_i^N(S)$ converges to the *exact* value of the American put $P_i(0, S)$ as N goes to infinity, as Carr (1998) did. Numerical results that will be shown later support this conjecture, but the proof of the conjecture is still unknown.

If we follow the argument in Carr (1998), we can easily derive the following equation for $P_i^k(\cdot)$:

$$\frac{1}{2}\sigma_i^2 S^2 P_{i,SS}^k(S) + rSP_{i,S}^k(S) - rP_i^k(S) + \lambda_i\left(P_j^k(S) \quad - P_i^k(S)\right) = \beta_N\left(P_i^k(S) - P_i^{k-1}(S)\right),$$
$$i \neq j, \quad j \in \{H, L\}, \quad \text{for} \quad S > \underline{S}_i^k.$$

Let $\underline{S}_M^k \equiv \max\left\{\underline{S}_H^k, \underline{S}_L^k\right\}$ and $\underline{S}_m^k \equiv \min\left\{\underline{S}_H^k, \underline{S}_L^k\right\}$. We use the same notation n_l's and ξ_l's$(l = 1, 2, 3, 4)$ as in the previous section with $\beta = \beta_N$.

The problem is to find $P_i^k(S)$ satisfying

$$\frac{1}{2}\sigma_i^2 S^2 P_{i,SS}^k(S) + rSP_{i,S}^k(S) - rP_i^k(S) + \lambda_i\left(P_j^k(S) - P_i^k(S)\right) = \beta_N\left(P_i^k(S) - P_i^{k-1}(S)\right), \quad i \neq j$$

and six boundary conditions

$$\lim_{S\uparrow\infty} P_i^k(S) = 0, \quad \lim_{S\downarrow\underline{S}_i^k} P_i^k(S) = K - \underline{S}_i^k, \quad \lim_{S\downarrow\underline{S}_i^k} P_{i,S}^k(S) = -1.$$

Theorem 4.3. Suppose that Assumption I is valid and the determinant of $\Phi(1)$, defined in (6.13), is not zero. Then $P_m^k(S)$ and $P_M^k(S)$ for $k = 1, 2, \ldots, N$ take the following form:

(i) In the region $S > \underline{S}_M^k$, for two constants C_h^k's$(h = 1, 2)$

$$\begin{cases} P_m^k(S) = \sum_{h=1}^{4}\left(C_h^k + u_h^k(S)\right)S^{n_h} \\ P_M^k(S) = \sum_{h=1}^{4}\xi_h\left(C_h^k + u_h^k(S)\right)S^{n_h} \end{cases} \tag{4.1}$$

where $u_h^k(\cdot)$'s are defined by (6.15) *in Appendix and*

$$C_h^k = -\lim_{S\to\infty} u_h^k(S) \quad \text{for} \quad h = 3, 4. \tag{4.2}$$

(ii) In the region $\underline{S}_m^k < S < \underline{S}_M^k$, for some constants D_1^k and D_2^k and m_1, m_2 defined in the previous section,

$$P_m^k(S) = D_1^k S^{m_1} + D_2^k S^{m_2}$$
$$-S^{m_1}\int_{\underline{S}_m^k}^{S}\frac{2\left(-\lambda_m K + \lambda_m t - \beta P_m^{k-1}(t)\right)}{(m_2 - m_1)\sigma_m^2 t^{m_1+1}}dt + S^{m_2}\int_{\underline{S}_m^k}^{S}\frac{2\left(-\lambda_m K + \lambda_m t - \beta P_m^{k-1}(t)\right)}{(m_2 - m_1)\sigma_m^2 t^{m_2+1}}dt,$$

and

$$P_M^k(S) = K - S.$$

Theorem 4.4. Suppose that Assumption I is valid. $\underline{S}_L^k > \underline{S}_H^k$ and $P_H^k(S) \geq P_L^k(S)$ for all $S > 0$. The inequality is strict for $S > \underline{S}_H^k$.

5. Implementation and numerical results

In this section, we will show numerical results. First, we will propose the Richardson extrapolation together with the values of American puts with a random expiration time studied in the previous section as an approximation to the American option with finite expiry.

We will compare the solutions obtained from this method with the solutions obtained by pentanomial trees in Bollen (1998) for a certain set of parameter values. Second, we will illustrate behavior of American put values in a regime-switching model with a parameter change by using numerical solutions obtained from the Richardson extrapolation method.

5.1 Richardson extrapolation with randomized put values

We first explain the Richardson extrapolation method. As in the previous section, let $P_i(0, S)$ denote the exact American Put value with time to expiration equal to $T < \infty$ at time 0 when the stock price is equal to S and regime is $i \in \{H, L\}$. And let $P_i^N(S;N)$ the value of an American put with an Erlang-distributed random time to expiration with $\beta = N/T$. Then, as in Geske and Johnson (1984), Broadie and Detemple (1996) and Carr (1998), the Richardson extrapolation for $M \geq 2$, $\tilde{P}_i^M(0,S)$, is defined as

$$\tilde{P}_i^M(0,S) \equiv \sum_{N=1}^{M} \frac{(-1)^{M-N} N^M}{N!(M-N)!} P_i^N(S;N).$$

For example, a three-point Richardson extrapolation is represented by

$$\tilde{P}_i^3(0,S) = \frac{1}{2}P_i^1(S;1) - 4P_i^2(S;2) + \frac{9}{2}P_i^3(S;3).$$

We conjecture that $\tilde{P}_i^M(0,S)$, for a sufficiently large M, is a good approximation to $P_i(0, S)$, namely,

$$P_i(0,S) \approx \tilde{P}_i^M(0,S),$$

for a sufficiently large M.

Table 1 exhibits the value of an American put calculated by the pentanomial-tree method with 10^3 time steps and those calculated by the method in this paper with three-point Richardson extrapolation. The default parameters are $r = 0.1$, $K = 1$, $T = 1$, $\sigma_L = 0.2$, $\lambda_L = 0.5$. It clearly shows that we can approximate the value of the American put quite accurately by the three-point Richardson extrapolation together with randomized put values (mostly errors are less than 0.0003).

Table 1. Comparison of the value of an American put option for a set of parameter values with default parameters $r = 0.1, K = 1, T = 1, \sigma_L = 0.2, \lambda_L = 0.5$

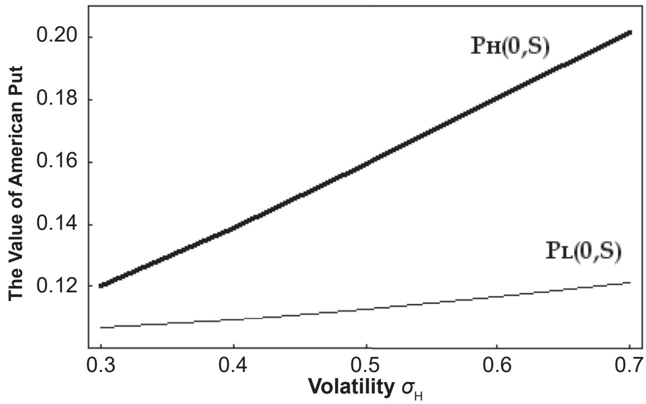
Parameters			Pentanomial tree		3-Point Richardson	
S	σ_H	λ_H	$P_H(0, S)$	$P_L(0, S)$	$\tilde{P}_H^3(0,S)$	$\tilde{P}_L^3(0,S)$
0.9	0.4	1.0	0.1483	0.1106	0.1483	0.1106
0.9	0.4	2.0	0.1390	0.1093	0.1393	0.1094
0.9	0.5	1.0	0.1738	0.1150	0.1737	0.1149
0.9	0.5	2.0	0.1594	0.1128	0.1597	0.1127
1.0	0.4	1.0	0.1015	0.0594	0.1014	0.0592
1.0	0.4	2.0	0.0904	0.0574	0.0905	0.0572
1.0	0.5	1.0	0.1293	0.0660	0.1292	0.0658
1.0	0.5	2.0	0.1128	0.0629	0.1131	0.0626

Note(s): The values in Columns 4 and 5 are obtained by the pentanomial-tree method in Bollen (1998) with 10^3 time steps and those in Columns 6 and 7 are obtained by our accelerated method with a three-point Richardson extrapolation
Source(s): Authors' own work

5.2 Characteristics of American put values

Figure 2 shows the value of an American put option as a function of volatility σ_H of the stock in state H for the following parameter values: $r = 0.1, K = 1, T = 1, \sigma_L = 0.2, \lambda_H = 2.0, \lambda_L = 0.5$ and $S = 0.9$. It shows the values in both the high-volatility state and low-volatility state increase as the high-state volatility σ_H increases.

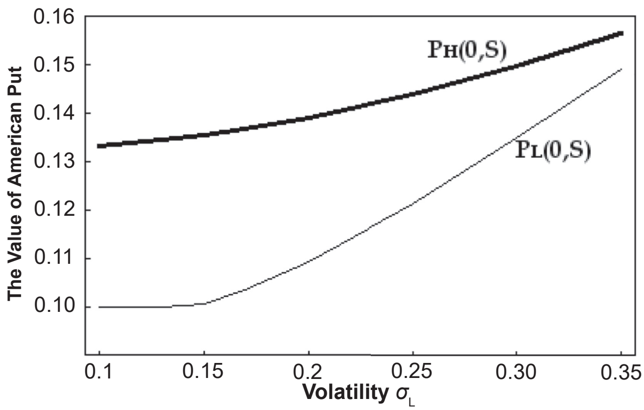
Figure 3 shows the value of an American put option as a function of volatility σ_L of the stock in state L for the following parameter values: $r = 0.1, K = 1, T = 1, \sigma_H = 0.4, \lambda_H = 2.0, \lambda_L = 0.5$ and $S = 0.9$. It shows the values in both the high-volatility state and low-volatility state increase as the low-state volatility σ_L increases. Put prices tend to increase as either the high-state volatility or the low-state volatility increases.



Note(s): We use the following parameter values: r = 0.1, K = 1, T = 1, σL = 0.2, λH = 2.0, λL = 0.5, and S = 0.9

Source(s): Authors' own work

Figure 2. Value of an American put option as a function of volatility σ_H of the stock in state H



Note(s): We use the following parameter values: r = 0.1, K = 1, T = 1, σH = 0.4, λH = 2.0, λL = 0.5, and S = 0.9

Source(s): Authors' own work

Figure 3. Value of an American put option as a function of volatility σ_L of the stock in state L

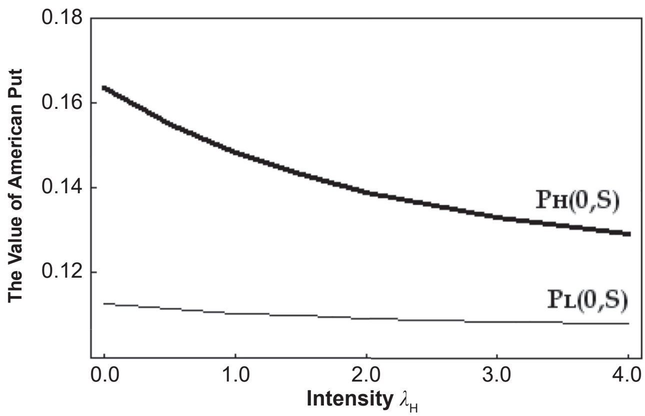
The results in Figures 2 and 3 are because there is a possibility that the future regime may differ from the current regime through regime shifts, thus the option price in a regime might be highly influenced by the volatility of the opposite regime. This implies, if a period incorporating regime shifts is considered as the lifetime of an American option, practitioners should assess the market value of options taking into account the volatility of a regime different from the present one.

Figure 4 shows the value of an American put option as a function of intensity parameter λ_H for the following parameter values $r = 0.1, K = 1, T = 1, \sigma_H = 0.4, \sigma_L = 0.2, \lambda_L = 0.5$ and $S = 0.9$. It shows the values in both the high-volatility state and low-volatility state decline as the intensity λ_H increases. Namely, as it is more likely for the high-volatility regime to change to the low-volatility regime, the option values in both the high-volatility state and low-volatility state tend to decline. Figure 5 shows the value of an American put option as a function of intensity parameter λ_L for the following parameter values $r = 0.1, K = 1, T = 1, \sigma_H = 0.4, \sigma_L = 0.2, \lambda_H = 2.0$ and $S = 0.9$. It shows the values in both the high-volatility state and low-volatility state increase as the intensity λ_L increases.

We conjecture the above comparative statics results illustrated by numerical examples are generally valid and leave their analytic proof as an open question.

6. Conclusion

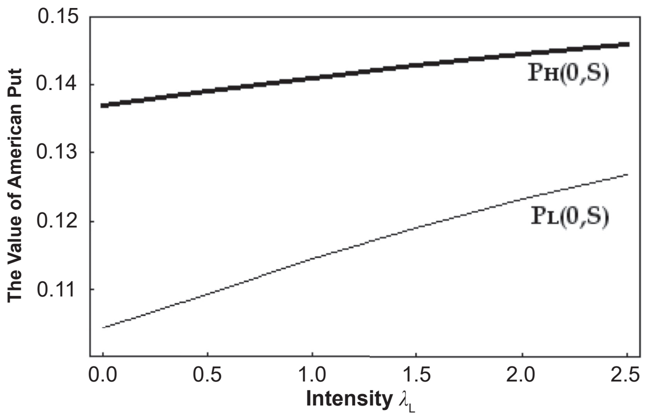
We introduce a novel approach to find the value of American puts with regime-switching volatility. We demonstrate that the value of an American put can be represented as the combination of the corresponding European put price and the premium attributed to the early exercise privilege. We also provide rigorous proof that the American put value constitutes a unique solution to the free boundary value problem. Furthermore, we show that perpetual American put options can exhibit distinct characteristics depending on the prevailing volatility regime. Specifically, we establish that perpetual put options command a higher price during periods of high-volatility compared to low-volatility phases. Additionally, we reveal that the optimal exercise boundary is situated at a lower threshold in high-volatility regimes, as opposed to low-volatility regimes. These findings



Note(s): We use the following parameter values r = 0.1, K = 1, T = 1, σH = 0.4, σL = 0.2, λL = 0.5, and S = 0.9

Source(s): Authors' own work

Figure 4. Value of an American put option as a function of intensity parameter λ_H



Note(s): We use the following parameter values r = 0.1, K = 1, T = 1, σH = 0.4, σL = 0.2, λH = 2.0, and S = 0.9
Source(s): Authors' own work

Figure 5. Value of an American put option as a function of intensity parameter λ_L

contribute valuable insights into the dynamics of option pricing in the context of fluctuating volatility regimes.

Moreover, we develop an analytical framework to describe American puts with an Erlang-distributed random-time horizon, which allows us to propose a numerical technique for approximating the value of American puts with finite expiry. We also show that a combined approach involving randomization and Richardson extrapolation can be a numerical algorithm for estimating American put prices with finite expiry.

Notes

1. Yi (2008) shows the same result with ours.
2. Analysis in this paper can be easily extended to the case where the stock pays cash dividends at a constant rate.
3. The probability measure Q is equivalent to P on $(\Omega, \mathcal{F}_T^{B,\varphi_H,\varphi_L})$ for $T < \infty$, where $\mathcal{F}_T^{B,\varphi_H,\varphi_L}$ is a σ-algebra generated by $B(s)$, $\varphi H(s)$, $\varphi L(s)$ for $s \in [0, T]$. But in an infinite horizon P and Q are not equivalent. See Section 1.7, Karatzas and Shreve (1998).
4. We need to enlarge the filtration $\{\mathcal{F}_t\}$ to make the Poisson process progressively measurable. We will assume such a modification has been done to the filtration.
5. The two equations are very lengthy and we do not report them in the paper. They are available from the authors upon request.

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Appendix

We provide proof of the results in this appendix.

The Snell Envelope. Let us define

$$Y(t) \equiv e^{-rt}(K-S(t))^{+}$$

By Theorem D.7 in Appendix of Karatzas and Shreve (1998) (Hereafter we will refer to it as "KS"), there is a Q-supermartingale $\{\xi(t)\colon 0 \le t \le T\}$ with right continuous and left limit (RCLL) paths such that

$$\xi(t) \ge Y(t) \quad \text{for all } t \in [0,T]$$

almost surely and for a stopping time $v \in [0, T]$

$$\xi(v) = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_{v,T}} E^{Q}[Y(\tau) |\ \mathcal{F}_v] \quad a.s.$$

where $\mathcal{S}_{v,T}$ is the set of stopping times between v and T. We know, in particular, $\xi(0) = \sup_{\tau \in \mathcal{S}_{0,T}} E^{Q}[Y(\tau)] = P_i(T,S)$. By Theorem D.12 in KS, with the stopping time

$$\tau^* \equiv \{t \in [0,T) : \xi(t) = Y(t)\} \wedge T,$$

we have $\xi(0) = E^Q Y(\tau^*)$.

Proof of Proposition 2.1. The proof is a modification of the proof of Theorem 5.3, Chapter 2 in KS. A self-financing process dominating the American option payoff satisfies the following inequality: there exists some portfolio process $(\pi(t), \pi_H(t), \pi_L(t))$, $t \in [0, T]$ and $\gamma \in R$ such that for every $\tau \in \mathcal{S}_{0,T}$

$$Y(\tau) \leq \gamma + \int_0^\tau \frac{1}{S(u)}\pi(u)\sigma(u)d\tilde{B}(u) + \int_0^T \frac{1}{S_H(u)}\pi_H(u)d\tilde{\varphi}_H(u) + \int_0^T \frac{1}{S_L(u)}\pi_L(u)d\tilde{\varphi}_L(u), \tag{6.1}$$

where $\tilde{\varphi}_i(t) \equiv \varphi_i(t) - \lambda_i t$ and $S_i(t)$ is the price of Θ_i at time i for $i \in \{H, L\}$ (by market completeness assumption Θ_i is tradable in the market).

Let

$$V_{\min} \equiv \inf\{\gamma \in \mathbb{R} : \text{there exists a portfolio process } (\pi(t), \pi_H(t), \pi_L(t)), t \in [0,T], \text{satisfying (6.1)}\}.$$

That is, $V_{\min}$ is the smallest initial wealth required such that there exists self-financing strategy whose wealth process satisfies (i) and (ii) in the proposition.

Theorem D.13 in KS asserts that $\xi(\cdot) = M(\cdot) - \Lambda(\cdot)$, where $M(\cdot)$ is a uniformly integrable RCLL martingale under Q and $\Lambda(\cdot)$ is an adapted, continuous, nonincreasing process with $\Lambda(0) = \Lambda(\tau^*) = 0$ a.s. Because of the assumption of market completeness, the $\mathcal{F}_T$-measurable random variable $B \equiv S(T)M(T)$ can be self-financed, i.e. by the martingale representation theorem for a process driven by a Brownian motion and independent Poisson processes (see Theorem 12.33 and 12.35 of Elliot (1980)), there is a triplet $(\hat{\pi}(\cdot), \hat{\pi}_H(\cdot), \hat{\pi}_L(\cdot))$ satisfying

$$\begin{aligned} M(T) = \xi(0) &+ \int_0^T \frac{1}{S(u)}\hat{\pi}(u)\sigma(u)d\tilde{B}(u) + \int_0^T \frac{1}{S_H(u)}\hat{\pi}_H(u)d\tilde{\varphi}_H(u) \\ &+ \int_0^T \frac{1}{S_L(u)}\hat{\pi}_L(u)d\tilde{\varphi}_L(u), \end{aligned}$$

where $\tilde{\varphi}_i(t) \equiv \varphi_i(t) - \lambda_i t$ and $S_i(t)$ is the price of Θ_i at time i for $i \in \{H, L\}$.

Taking conditional expectations with respect to $\mathcal{F}_t$ in the above equation, we have for $t \in [0, T]$

$$\begin{aligned} Y(t) \leq \xi(t) &= M(t) - \Lambda(t) \\ &= \xi(0) - \Lambda(t) + \int_0^t \frac{1}{S(u)}\hat{\pi}(u)\sigma(u)d\tilde{B}(u) + \int_0^t \frac{1}{S_H(u)}\hat{\pi}_H(u)d\tilde{\varphi}_H(u) + \int_0^t \frac{1}{S_L(u)}\hat{\pi}_L(u)d\tilde{\varphi}_L(u) \\ &\leq \xi(0) + \int_0^t \frac{1}{S(u)}\hat{\pi}(u)\sigma(u)d\tilde{B}(u) + \int_0^t \frac{1}{S_H(u)}\hat{\pi}_H(u)d\tilde{\varphi}_H(u) + \int_0^t \frac{1}{S_L(u)}\hat{\pi}_L(u)d\tilde{\varphi}_L(u). \end{aligned}$$

Therefore,

$$V_{\min} \leq \xi(0) = P_i(0, S),$$

Suppose that there exists a self-financing strategy whose wealth process $X(t)$ satisfies (i) and (ii) in the statements of the proposition. For any stopping time $\tau \in \mathcal{S}_{0,T}$

$$\begin{aligned} Y(\tau) \leq X(0) &+ \int_0^\tau \frac{1}{S(u)}\hat{\pi}'(u)\sigma(u)d\tilde{B}(u) + \int_0^\tau \frac{1}{P_H(u)}\hat{\pi}'_H(u)d\tilde{\varphi}_H(u) \\ &+ \int_0^\tau \frac{1}{P_L(u)}\hat{\pi}'_L(u)d\tilde{\varphi}_L(u), \end{aligned}$$

where $(\widehat{\pi}'(\cdot), \widehat{\pi}'_H(\cdot), \widehat{\pi}'_L(\cdot))$ is a self-financing portfolio for the process $X(\cdot)$. Taking expectation in the above equation we know

$$E^Q Y(\tau) \le X(0), \quad \forall \tau \in \mathcal{S}_{0,T}.$$

Therefore, $P_i(0, S) = \xi(0) \le X(0)$ and this implies that

$$P_i(0, S) \le V_{\min}.$$

This completes the proof. ♠

Proof of Lemma 2.2. The proof is essentially the same as that of Proposition 7.1 and Lemma 7.4, Chapter 2 in KS. Therefore, we establish only the proof of the uniform continuity in T here.

Let us define, for $i \in \{H, L\}$,

$$\psi_i(t) \equiv E^Q\left[\max_{0\le s\le t}\{1 - e^{-rs}H_i(s)\}^+\right],$$

$$H_i(s) \equiv \exp\left(rs - \frac{1}{2}\int_0^s \sigma^2(u)du + \int_0^s \sigma(u)d\tilde{B}_u\right) \quad \text{when} \quad \sigma(0) = \sigma_i,$$

and

$$\psi(t) \equiv \psi_H(t) \vee \psi_L(t).$$

Then by the bounded convergence theorem, $\lim_{t\downarrow 0}\psi(t) = 0$. Suppose that $\sigma(0) = \sigma_i$. For given $S_0 = S \in [0, \infty)$, if we let $0 \le T_1 \le T_2$ and

$$\tau_2 = \inf\{t \in [0, T_2) \,|\, \xi(t) = e^{-rt}(K - S\,H_i(t))^+\} \wedge T_2, \quad \tau_1 = \tau_2 \wedge T_1,$$

then by the fact $S(t) = S\,H(t)$, we obtain

$$\begin{aligned}
0 &\le P_i(T_2, S) - P_i(T_1, S) \le E^Q\left[e^{-r\tau_2}(K - S(\tau_2))^+ - e^{-r\tau_1}(K - S(\tau_1))^+\right] \\
&\le E^Q\left[(e^{-r\tau_1}S(\tau_1) - e^{-r\tau_2}S(\tau_2))^+\right] \\
&\le E^Q\left[e^{-r\tau_1}S(\tau_1)\cdot E^Q\left[\left(1 - \min_{T_1\le t\le T_2}\left\{\exp\left(-\frac{1}{2}\int_{T_1}^t \sigma^2(u)du + \int_{T_1}^t \sigma(u)d\tilde{B}_u\right)\right\}\right)^+ \middle| \mathcal{F}_{T_1}\right]\right] \\
&\le E^Q[e^{-r\tau_1}S(\tau_1)]\cdot\psi(T_2 - T_1) \le S\psi(T_2 - T_1).
\end{aligned}$$

Thus, $P_i(T, S)$ is uniformly continuous in T for all $S \in [0, \infty)$. ♠

We need the following lemmas in order to prove Theorem 2.3 and Proposition 2.5.

Lemma 6.1. $(P_H(u, S), P_L(u, S))$ is a solution to the initial-boundary value problem

$$\begin{aligned}
\mathcal{L}f_i(u, S) &= 0, \quad S > \underline{S}_i(u), \quad u \in [0, \infty) \\
f_i\left(u, \underline{S}_i(u)\right) &= K - \underline{S}_i(u), \quad u \in [0, \infty), \\
f_i(0, S) &= (K - S)^+, \quad S \ge \underline{S}_i(0), \\
\lim_{S\to\infty}\max_{0\le u\le T}|f_i(u, S)| &= 0 \quad \text{for all } T < \infty,
\end{aligned}$$

for $i \in \{H, L\}$. In particular, the partial derivatives $P_{i,SS}$, $P_{i,S}$, $P_{i,t}$ exist and are continuous in $\mathcal{C}_i$ for $i \in \{H, L\}$.

Proof. The proof is a modification of the proof of Theorem 7.6 Chapter 2 in KS.

Consider the following *initial-boundary value problem*: for $i \in \{H, L\}$

$$\frac{\sigma_i^2}{2} S^2 f_{i,SS}(u,S) + rSf_{i,S}(u,S) - rf_i(u,S) - \lambda_i f_i(u,S) - f_{i,u}(u,S) = -\lambda_i P_j(u,S), \quad \text{in } \mathcal{R}$$

$$f_i = P_i, \qquad \text{on } \partial_0 \mathcal{R},$$

where $\mathcal{R} = (t_1, t_2) \times (S_1, S_2) \subset \mathcal{C}_i$ and $\partial_0 \mathcal{R} = \partial_0 \mathcal{R} - \{t_2\} \times (S_1, S_2)$.

We know by Lemma 2.2 $P_i (i \in \{H, L\})$ is uniformly Hölder continuous and Theorem 7, Chapter 3 in Friedman (1964) implies that there exists a unique classical solution to the above initial-boundary value problem.

We show that $f_i = P_i$ on $\mathcal{R}$. Suppose that $\sigma(0) = \sigma_i$ and $j \in \{H, L\}, j \neq i$,. Let (t_0, S_0) be an element in $\mathcal{R}$ and consider a stopping time $\tau \in \mathcal{S}_{0,t_0-t_1}$ defined by

$$\tau \equiv \inf\{\theta \in [0, t_0 - t_1) : (t_0 - \theta, S(\theta)) \in \partial_0 \mathcal{R}\} \wedge \inf\{\theta \in [0, t_0 - t_1) : \sigma(\tau) = \sigma_j\} \wedge (t_0 - t_1)$$

where $S(\theta)$ is the price of the stock at time θ with $S(0) = S_0$ and the process

$$N(\theta) \equiv e^{-r\theta} f_i(t_0 - \theta, S(\theta)) \chi_{\{\sigma(\theta)=\sigma_i\}} + e^{-r\theta} P_j(t_0 - \theta, S(\theta)) \chi_{\{\sigma(\theta)=\sigma_j\}}, \quad \theta \in [t_0 - t_1].$$

By Ito's rule $N(\cdot \wedge \tau)$ is a bounded P_0-martingale, and thus

$$f(t_0, S_0) = N(0) = E^Q N(\tau) = \sum_{k \in \{H,L\}} E^Q \left[e^{-r\tau} P_k(t_0 - \tau, S(\tau)) \chi_{\{\sigma(\tau)=\sigma_k\}} \right].$$

But $(t_0 - \tau, S(t)) \in \mathcal{C}_i$ implies

$$\tau \leq \tau_S \equiv \inf\left\{\theta \in [0, t_0) : \ \xi(t_0 - \theta) = (K - S(\theta))^+\right\} \wedge t_0$$

Now the proof of Proposition 2.1 implies that the stopped process

$$\left\{ \left(e^{-r(t \wedge \tau_S)} \xi(T - (t \wedge \tau_S)), \ \mathcal{F}_t \right) \ : \ 0 \leq t \leq T \right\} \tag{6.2}$$

is a Q-martingale. Now the optional sampling theorem and equation (6.2) yield

$$\sum_{k \in \{H,L\}} E^Q \left[e^{-r\tau} P_k(t_0 - \tau, S(\tau)) \chi_{\{\sigma(\tau)=\sigma_k\}} \right] = P_i(t_0, S_0).$$

Thus, f_i and P_i agree on $\mathcal{R}$ and hence $P_{i,SS}$, $P_{i,S}$ and $P_{i,t}$ exist and continuous in $\mathcal{C}_i$.

We now show the last limit in the lemma. Let $T \in (0, \infty)$ and for $(t, S) \in [0, T] \times (0, \infty)$ define

$$\tau_S \equiv \inf\left\{\theta \in [0, t) \ : \ P_i(t - \theta, S(\theta)) = (K - S(\theta))^+\right\} \wedge t$$

where $S(\theta)$ is the price of the stock at time θ with $S(0) = S$. Notice that $P_i(t, S) = E^Q[e^{-r\tau_S} (K - S(\tau_S))^+]$. Set

$$\rho_S \equiv \{\theta \in [0, \infty) : S(\theta) \leq K\}.$$

Then,

$$\begin{aligned} 0 \leq P_i(t,S) &\leq KE^Q\left[\chi_{\{\rho_S \leq t\}} e^{-r\rho_S}\right] + E^Q\left[\chi_{\{\rho_S > t\}} e^{-rt}(K - S(t))^+\right] \\ &\leq KQ[\rho_S \leq T]. \end{aligned}$$

Since $\lim_{S\to\infty} Q[\rho_S \leq T] = 0$, the last limit in the lemma is valid. ♠

Lemma 6.2. The Snell envelope $\xi(t)$ has the following decomposition

$$\xi(t) = M(t) - \Lambda(t), \quad t \in [0,T], \tag{6.3}$$

where

$$\begin{aligned} M(t) \equiv\ & P_i(T,S) + \sum_{k\in\{H,L\}} \int_0^t e^{-ru} S(u) P_{k,S}(T-u, S(u)) \sigma_k \chi_{\{\sigma(u)=\sigma_k\}} d\tilde{B}(u) \\ & + \sum_{\substack{k,j\in\{H,L\}, \\ j\neq k}} \int_0^t e^{-ru} (P_j(T-u, S(u)) - P_k(T-u, S(u)) \chi_{\{\sigma(u)=\sigma_k\}} d\tilde{\varphi}_k(u), \end{aligned}$$

and

$$\Lambda(t) \equiv \int_0^t rKe^{-ru} \chi_{\left\{S(u)<\underline{S}_H \& \sigma(u)=\sigma_H\right\} \cup \left\{S(u)<\underline{S}_L \& \sigma(u)=\sigma_L\right\}} du.$$

Proof. The proof is a modification of the proof of Theorem 7.9, Chapter 2 in KS. Let $\zeta : \mathbb{R}^2 \to [0,\infty)$ be a C^∞ function integrating to 1 and having support in $[0,1]^2$. For $k \in \{H,L\}$ and $\epsilon > 0$, define

$$P_k^\epsilon(t,S) \equiv \int_0^\infty \int_0^\infty P_k(t+\epsilon u, S+\epsilon v)\zeta(u,v)dudv.$$

Then, $P_k^\epsilon(t,S)$ is of class C^∞ on $(0,\infty)^2$. And by the same calculation as in the proof of Theorem 7.9 in KS, we know

$$\begin{aligned} P_{k,S}^\epsilon(t,S) &= \int_0^\infty \int_0^\infty P_{k,S}((t+\epsilon u, S+\epsilon v)\zeta(u,v)dudv \\ P_{k,SS}^\epsilon(t,S) &= \int_0^\infty \int_0^\infty P_{k,SS}((t+\epsilon u, S+\epsilon v)\zeta(u,v)dudv \\ P_{k,t}^\epsilon(t,S) &= \int_0^\infty \int_0^\infty P_{k,t}((t+\epsilon u, S+\epsilon v)\zeta(u,v)dudv \end{aligned}$$

These formulas show that P_k^ϵ and $\mathcal{L}P_k^\epsilon$ are bounded on compact subsets of $(0,\infty)^2$ and

$$\begin{aligned} P_{k,S}(t,S) &= \lim_{\epsilon\downarrow 0} P_{k,S}^\epsilon(t,S), \\ \mathcal{L}P_k(t,S) &= \lim_{\epsilon\downarrow 0} \mathcal{L}P_k^\epsilon(t,S), \quad \forall (t,S) \in (0,\infty)^2, \ S \neq \underline{S}_i(t). \end{aligned}$$

According to generalized Ito's rule,

$$\begin{aligned}
&\sum_{k\in\{H,L\}} e^{-rt}P_k^{\epsilon}(T-t,S(t))\chi_{\{\sigma(u)=\sigma_k\}} \\
&= P_i(T,S) + \sum_{k\in\{H,L\}}\int_0^t e^{-ru}\mathcal{L}P_k^{\epsilon}(T-u,S(u))\chi_{\{\sigma(u)=\sigma_k\}}du \\
&+ \sum_{k\in\{H,L\}}\int_0^t e^{-ru}S(u)P_{k,S}^{\epsilon}(T-u,S(u))\sigma_k\chi_{\{\sigma(u)=\sigma_k\}}d\tilde{B}(u) \\
&+ \sum_{\substack{k,j\in\{H,L\},\\ j\neq k}}\int_0^t e^{-ru}\left(P_j^{\epsilon}(T-u,S(u)) - P_k^{\epsilon}(T-u,S(u))\chi_{\{\sigma(u)=\sigma_k\}}\right)d\tilde{\varphi}_k(u),
\end{aligned} \tag{6.4}$$

for $t \in [0, T]$.

For each $u \in (0, T)$, we have

$$Q\left[\left\{S(u) = \underline{S}_H \ \&\sigma(u) = \sigma_H\right\} \cup \left\{S(u) = \underline{S}_L \ \&\sigma(u) = \sigma_L\right\}\right] = 0.$$

Therefore, $\sum_{k\in\{H,L\}}\int_0^t e^{-ru}\mathcal{L}P_S(T-u,S(u))\chi_{\{\sigma(u)=\sigma_k\}}du$ is defined and equal to

$$\int_0^t e^{-ru}rK\chi_{\left\{S(u)<\underline{S}_H\&\sigma(u)=\sigma_H\right\}\cup\left\{S(u)<\underline{S}_L\&\sigma(u)=\sigma_L\right\}}du,$$

a.s.

Letting $\epsilon \downarrow 0$ in equation (6.4), we obtain (6.3), for $t \in [0, T)$ and then for $t = T$ by letting $t \uparrow T$. The process $M(\cdot)$ is a martingale because $-1 \leq P_{k,S}(t, S) \leq 0$ and $E^Q\left[\int_0^T S(u)^2 du\right] < \infty$. ♠

Proof of Theorem 2.3. Suppose that $\sigma(0) = \sigma_i (i \in \{H, L\})$. By Lemma 6.2, we have

$$\begin{aligned}
p_i(T,S) &= E^Q\left[e^{-rT}(K-S(T))^+\right] = E^Q[\xi(T)] \\
&= E^Q[M(T)] - E^Q[\Lambda(T)] = \xi(0) - E^Q[\Lambda(T)] \\
&= P_i(T,S) - E^Q[\Lambda(T)].
\end{aligned}$$

Therefore, the decomposition in the theorem follows. ♠

Proof of Proposition 2.5. The proof is a modification of the proof of Theorem 7.12, Chapter 2 in KS. $(P_H(u, S), P_L(u, S))$ and $\left(\underline{S}_H(u), \underline{S}_L(u)\right)$ solve Problem 2.4 by Lemma 6.1.

Suppose $(f_H(u, S), f_L(u, S))$ is a solution to Problem 2.4 together with $(\mathcal{D}_H(u), \mathcal{D}_L(u))$.

Fix $(T, S) \in [0, \infty)^2$ and assume that $S(0) = S$ and $\sigma(0) = \sigma_i (i \in \{H, L\})$. Let us define a process $\xi^f(t)$ by letting

$$\xi^f(t) \equiv \sum_{k\in\{H,L\}} e^{-rt}f_k(T-t,S(t))\chi_{\{\sigma(t)=\sigma_k\}}.$$

By using the same argument as in the proof of Lemma 6.2, we obtain

$$\xi^f(t) = M^f(t) - \Lambda^f(t)$$

where $M^f(\cdot)$ is a local martingale defined according as

$$
\begin{aligned}
M^f(t) \quad \equiv \quad & f_i(T,S) + \sum_{k\in\{H,L\}} \int_0^t e^{-ru} S(u) f_{k,S}(T-u,S(u))\sigma_k \chi_{\{\sigma(u)=\sigma_k\}} d\tilde{B}(u) \\
& + \sum_{\substack{k,j\in\{H,L\},\\ j\neq k}} \int_0^t e^{-ru}\Big(f_j(T-u,S(u)) - f_k(T-u,S(u))\chi_{\{\sigma(u)=\sigma_k\}} d\tilde{\varphi}_k(u),
\end{aligned}
$$

and $\Lambda^f(\cdot)$ is nondecreasing and defined as

$$
\Lambda^f(t) \equiv \int_0^t e^{-ru} rK\chi_{\{S(u)<\mathcal{D}_H \& \sigma(u)=\sigma_H\}\cup\{S(u)<\mathcal{D}_L \& \sigma(u)=\sigma_L\}} du.
$$

Let $(\tau_n)_{n=1}^{\infty}$ be a sequence of stopping times with $\tau_n \uparrow T$ almost surely and such that $\{M^f(t\wedge\tau_n): 0 \le t \le T\}$ is a Q-martingale. For any stopping time $\tau \in \mathcal{S}_{0,T}$ we have

$$
E^Q\left[\xi^f(\tau\wedge\tau_n)\right] = f_i(T,S) - E^Q\Lambda^f(\tau\wedge\tau_n).
$$

By the fact that $\lim_{S\uparrow\infty} \max_{0\le t\le T} |f_k(u,S)| = 0$, $k \in \{H,L\}$, the function f_k is bounded on $[0,T]\times[0,\infty)$, so by taking limit $n\to\infty$ in the above equation the dominated convergence theorem implies

$$
E^Q\left[\xi^f(\tau)\right] = f_i(T,S) - E^Q\Lambda^f(\tau), \quad \forall \tau \in \mathcal{S}_{0,T}. \tag{6.5}
$$

From the fact $f_k(u,S) \ge (K-S)^+$ for $k \in \{H,L\}$ and $(u,S)\in[0,\infty)^2$ and the nonnegativity of Λ^f, we have

$$
E^Q\left[e^{-r\tau}(K-S(\tau))^+\right] \le f_i(T,S), \quad \forall \tau \in \mathcal{S}_{0,T}.
$$

Therefore, $f_i(T,S) \ge P_i(T,S)$.

Let us now show the reverse inequality. Let

$$
\tau'_S \equiv T \wedge \inf\{t \ge 0: \ (S(t)\le\mathcal{D}_H \& \sigma(t)=\sigma_H) \text{ or } (S(t)\le\mathcal{D}_L \& \sigma(t)=\sigma_L)\}
$$

Then, we have $\xi^f(T-\tau'_S) = (K-S(\tau'_S))^+$ and $\Lambda^f(\tau'_S) = 0$, so equation (6.5) implies $E^Q[e^{-r\tau'_S}(K-S(\tau'_S))^+] = f_i(T,S)$. This implies $f_i(T,S) \le P_i(T,S)$, and therefore, $f_i(T,S) = P_i(T,S)$.

To show that $\mathcal{D}_i = \underline{S}_i$, it is enough to show the two open sets $\mathcal{C}_i^0$ and $\mathcal{D}_i$ are equal, where $\mathcal{C}_i^0 \equiv \left\{(t,S)\in(0,\infty)^2 : S > \underline{S}_i(t)\right\}$ and $\mathcal{D}_i \equiv \{(t,S)\in(0,\infty)^2 : S > \mathcal{D}_i(t)\}$. For $(t,S)\in\mathcal{C}_i^0$, we have $\mathcal{L}P_i(t,S) = 0$ which means that $(t,S)\in\overline{\mathcal{D}}_i$. Therefore $\mathcal{C}_i^0 \subseteq \mathcal{D}_i$ since $\mathcal{D}_i$ is open. The other inclusion $\mathcal{D}_i \subseteq \mathcal{C}_i^0$ can be proved by the same argument. ♠

Proof of Proposition 3.2. We can extend Lemma 6.1 to the case $T=\infty$ by considering a system of the second-order ordinary differential equations instead of parabolic partial differential equations. Namely, for $i \in \{H,L\}$

$$
\frac{\sigma_i^2}{2} S^2 f_{i,SS}(S) + rSf_{i,S}(S) - rf_i(S) - \lambda_i f_i(S) = -\lambda_i P_j(S), \quad \text{in } \mathcal{R}
$$

$$
f_i = P_i, \qquad \text{on } \partial\mathcal{R},
$$

where $\mathcal{R} = (S_1,S_2)\subset\mathcal{C}_i$ and $\partial\mathcal{R} = \{S_1,S_2\}$. We know by the theory of ordinary differential equations that there exists a unique classical solution to the above boundary value problem. Then, the proof of the extended lemma follows in exactly the same way as in the proof of Lemma 6.1.

Now $(P_H(S), P_L(S))$ and $\left(\underline{S}_H, \underline{S}_L\right)$ is a solution to Problem 3.1 by the extended lemma.

Conversely, a solution $(f_H(S), f_L(S))$ and $(\mathcal{D}_H, \mathcal{D}_L)$ to Problem 3.1 satisfies $f_i(S) = P_i(S)$ and $\mathcal{D}_i = \underline{S}_i$ for $i \in \{H, L\}$ by Theorem 3.1 in Guo and Zhang (2004). ♠

Proof of Lemma 3.3. We first try to find a solution of the form $P_H(S) = S^n, P_L(S) = \xi S^n$ to (3.1), where n and ξ are some complex numbers. A straightforward calculation shows that if n and ξ satisfy the following system of algebraic equations then $P_H(S) = S^n$, $P_L(S) = \xi S^n$ are solutions to (3.1):

$$\begin{cases} \frac{1}{2}\sigma_H^2 n(n-1) + rn - r - \lambda_H + \lambda_H \xi = 0 \\ \xi\left(\frac{1}{2}\sigma_L^2 n(n-1) + rn - r - \lambda_L\right) + \lambda_L = 0. \end{cases} \tag{6.6}$$

Now n satisfies equations (6.6) if and only if it satisfies the fourth-order equation:

$$(n-1)\left[\sigma_H^2\sigma_L^2 n^3 + \left(2r\sigma_H^2 + 2r\sigma_L^2 - \sigma_H^2\sigma_L^2\right)n^2 + 2\left(2r^2 - r\sigma_H^2 - r\sigma_L^2 - \lambda_H\sigma_L^2 - \lambda_L\sigma_H^2\right)n \right. \\ \left. - 4r(r + \lambda_H + \lambda_L)\right] = 0. \tag{6.7}$$

We will now show that all the roots of equation (6.7) are real and exactly two of them are negative. Obviously, $n = 1$ is a one positive root of equation (6.7). Let

$$I(n) \equiv \sigma_H^2\sigma_L^2 n^3 + \left(2r\sigma_H^2 + 2r\sigma_L^2 - \sigma_H^2\sigma_L^2\right)n^2 + 2\left(2r^2 - r\sigma_H^2 - r\sigma_L^2 - \lambda_H\sigma_L^2 - \lambda_L\sigma_H^2\right)n - 4r(r \\ + \lambda_H + \lambda_L).$$

Then,

$$I(0) = -4r(r + \lambda_1 + \lambda_2) < 0, \quad I\left(-\frac{2r}{\sigma_L^2}\right) = \frac{4r\lambda_2\left(\sigma_H^2 - \sigma_L^2\right)}{\sigma_L^2} > 0.$$

Therefore, there exist two distinct real negative roots and two (possibly equal) real positive roots of equation (6.7).

Note that among the solutions of the form $P_H(S) = S^n, P_L(S) = \xi S^n$, only those with negative n satisfy boundary conditions $\lim_{S\uparrow\infty} P_i(S) = 0$ for $i \in \{H, L\}$.

We know that the homogeneous system of equations (3.1) has a solution space of dimension 4. Suppose that the positive real roots to (6.7) are distinct. Then the basis to the solution space consists of pairs of functions $P_H(S) = S^n$, $P_L(S) = \xi S^n$ with n, ξ satisfying (6.6). Solutions to (3.1) satisfying the boundary conditions $\lim_{S\uparrow\infty} P_i(S) = 0$ for $i = H, L$ form a two-dimensional subspace whose basis consists of pairs of functions $P_H(S) = S^n, P_L(S) = \xi S^n$ with $n < 0$ and ξ satisfying (6.6). Therefore, a general solution takes the form in equation (3.3). A straightforward extension of the proof shows that the result is also valid for the case where the positive real roots to (6.7) are both equal to 1.

It is easy to show that $I\left(-\frac{2r}{\sigma_H^2}\right) < 0$. This proves inequalities for n_1 and n_2 in (3.4).

Let $J(n) \equiv \frac{1}{2}\sigma_H^2 n(n-1) + rn - r$. Then, it is easy to show that $J'(n) < 0$ for $n \le -\frac{2r}{\sigma_H^2}$ and $J\left(-\frac{2r}{\sigma_H^2}\right) = 0$. This proves inequalities $1 > \xi_1 > \xi_2$. Also, the first and second equations in (3.1) can be converted into

$$1 - \xi = (n-1)\left(\frac{1}{2}\sigma_H^2 n + r\right)\Big/\lambda_H, \quad 1 - \frac{1}{\xi} = (n-1)\left(\frac{1}{2}\sigma_L^2 n + r\right)\Big/\lambda_L,$$

respectively. Dividing the first equation by the second, we obtain

$$\xi = -\frac{\lambda_H\left(\frac{1}{2}\sigma_L^2 n + r\right)}{\lambda_L\left(\frac{1}{2}\sigma_H^2 n + r\right)},$$

and this fact together with $n_2 < -2r/\sigma_L^2 < n_1 < -2r/\sigma_H^2$ implies that $\xi_1 > 0 > \xi_2$.

The proof is now complete. ♠

Proof of Lemma 3.4.Let us assume $\underline{S}_m < S < \underline{S}_M$. Then, by (3.2) we know that $P_M(S) = K - S$. Therefore, $P_m(S)$ satisfies

$$\frac{1}{2}\sigma_m^2 S^2 P_{m,SS} + rSP_{m,S} - (r+\lambda_m)P_m + \lambda_m(K-S) = 0.$$

Now a solution to the above equation takes the form in equation (3.5), where m_1, m_2 are solutions to the characteristic equation

$$\frac{1}{2}\sigma_m^2 z^2 + \left(r - \frac{1}{2}\sigma_m^2\right)z - (r+\lambda_m) = 0.$$

Therefore, m_1, m_2 satisfy inequalities in (3.6). This completes the proof. ♠

Proof of Proposition 3.5. From the boundary conditions for $P_m(S)$ at $\underline{S}_m$, we obtain equation (3.9), and from the boundary conditions for $P_M(S)$ at $\underline{S}_M$, we get equation (3.8). Finally, by (3.9) and (3.8) the C^2-property of $P_m(S)$ at $\underline{S}_M$ is equivalent to equation (3.7). ♠

Proof of Corollary 3.6. (i) (3.6) and (3.9) yield the property for D_1 and D_2 immediately.

(ii) Equation (3.8) implies that both C_1 and C_2 are non-zero. ♠

Proof of Theorem 3.7. Assume that the converse is true, i.e. $\underline{S}_H \geq \underline{S}_L$.

By Lemma 2.2, the value of the American put is a convex function of the underlying stock price S (The lemma is stated for a put option with finite expiry, but the proof is valid also for a perpetual put).

Therefore, $P_{i,SS}(S) \geq 0$ for all $S > \underline{S}_i$ for $i = H, L$. Convexity of $P_L(S)$ and the boundary condition $P_{L,S}\left(\underline{S}_L\right) = -1$ implies that

$$P_H(S) = K - S \leq P_L(S) \qquad \text{for } \underline{S}_L \leq S \leq \underline{S}_H,$$

and

$$P_{L,S}\left(\underline{S}_H\right) - P_{H,S}\left(\underline{S}_H\right) \geq 0. \tag{6.8}$$

Therefore,

$$P_L\left(\underline{S}_H\right) - P_H\left(\underline{S}_H\right) \equiv l_1 \geq 0. \tag{6.9}$$

From (3.3), we know $\lim_{S\uparrow\infty} P_{H,SS}(S) = 0$, and hence there exists $l_2 > 0$ such that $0 \leq P_{H,SS}(S) \leq l_2$.

Let $g(S) \equiv P_L(S) - P_H(S)$. Then, for $S \geq \underline{S}_H$, we derive the following equation from (3.1), (3.2), (6.8) and (6.9)

$$\frac{1}{2}\sigma_L^2 S^2 g''(S) + rSg'(S) - (r+\lambda_H+\lambda_L)g(S) = \frac{1}{2}\left(\sigma_H^2 - \sigma_L^2\right)S^2 P_{H,SS}(S),$$

with boundary conditions $g\left(\underline{S}_H\right) = l_1,\ g'\left(\underline{S}_H\right) \geq 0,\ \lim_{S\uparrow\infty} g(S) = 0$. Let $L(S) \equiv \frac{\sigma_H^2 - \sigma_L^2}{\sigma_L^2} P_{H,SS}(S)$. Then $0 \leq L(S) \leq l_3$, for some constant l_3, since $0 \leq P_{H,SS}(S) \leq l_2$. Applying a standard argument for an Euler equation, a general solution takes the following form

$$g(S) = c_1 S^{k_1} + c_2 S^{k_2} - S^{k_1}\int_{\underline{S}_H}^{S} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_1-1}}dt + S^{k_2}\int_{\underline{S}_H}^{S} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_2-1}}dt$$

for some constants c_1 and c_2, where.

$k_1 \equiv \frac{r-\sigma_L^2/2+\sqrt{(r-\sigma_L^2/2)^2+2\sigma_L^2(r+\lambda_H+\lambda_L)}}{\sigma_L^2}$, $\quad k_2 \equiv \frac{r-\sigma_L^2/2-\sqrt{(r-\sigma_L^2/2)^2+2\sigma_L^2(r+\lambda_H+\lambda_L)}}{\sigma_L^2}$.

Note that $k_1 > 1$ and $k_2 < 0$.

By (3.3), we know that

$$\left|\int_{\underline{S}_H}^{\infty} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_1-1}}dt\right| < \infty.$$

From (3.3) and the fact that $n_1, n_2, k_2 < 0$, we know

$$\lim_{S\uparrow\infty} S^{k_2}\int_{\underline{S}_H}^{S} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_2-1}}dt = 0. \tag{6.10}$$

(6.10), the fact that $k_1 > 1$ and the boundary condition $\lim_{S\uparrow\infty} g(S) = 0$ force c_1 to satisfy the following

$$c_1 = \int_{\underline{S}_H}^{\infty} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_1-1}}dt.$$

Here, the improper integral in the right-hand side is finite because of equation (3.3) and the fact that $n_1, n_2 < 0$, $k_1 > 0$.

Therefore,

$$l_1 = g\left(\underline{S}_H\right) = \underline{S}_H^{k_1}\int_{\underline{S}_H}^{\infty} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_1-1}}dt + c_2\underline{S}_H^{k_2},$$

or equivalently,

$$c_2 = l_1\underline{S}_H^{-k_2} - \underline{S}_H^{k_1-k_2}\int_{\underline{S}_H}^{\infty} \frac{L(t)}{(k_2-k_1)\sigma_L^2 t^{k_1-1}}.$$

But,

$$g'\left(\underline{S}_H\right) = k_1 c_1 \underline{S}_H^{k_1-1} + k_2 c_2 \underline{S}_H^{k_2-1} = -\underline{S}_H^{k_1-1}\int_{\underline{S}_H}^{\infty} \frac{L(t)}{\sigma_L^2 t^{k_1-1}}dt + l_1 k_2 \underline{S}_H^{-1} < 0$$

where the last inequality follows from the fact that $L(t)$ is not identically zero for $S \geq \underline{S}_H$ because of Corollary 3.6 (ii). This is contradictory to the boundary condition $g'\left(\underline{S}_H\right) \geq 0$.

The proof is complete. ♠

Proof of Theorem 3.8. Let $h(S) \equiv P_H(S) - P_L(S)$. Then, by an argument similar to the proof of Theorem 3.7, we know that $h(S)$ satisfies the following equation for $S > \underline{S}_L$

$$\frac{1}{2}\sigma_H^2 S^2 h''(S) + rSh'(S) - (r+\lambda_H+\lambda_L)h(S) = \frac{1}{2}\left(\sigma_L^2 - \sigma_H^2\right)S^2 P_{L,SS}(S),$$

with boundary conditions $h\left(\underline{S}_L\right) \equiv l \geq 0$, $h'\left(\underline{S}_L\right) \geq 0$, $\lim_{S\uparrow\infty} h(S) = 0$. Let $L(S) \equiv \frac{\sigma_L^2-\sigma_H^2}{\sigma_H^2}P_{L,SS}(S)$. Then, $-\tilde{l} \leq L(S) \leq 0$ for some positive constant $\tilde{l}$.

Applying a standard argument, we know

$$h(S) = c_1 S^{k_1} + c_2 S^{k_2} - S^{k_1}\int_{\underline{S}_L}^{S}\frac{L(t)}{(k_2-k_1)\sigma_H^2 t^{k_1-1}}dt + S^{k_2}\int_{\underline{S}_L}^{S}\frac{L(t)}{(k_2-k_1)\sigma_H^2 t^{k_2-1}}dt \qquad (6.11)$$

for some constants c_1 and c_2, where

$$k_1 = \frac{r-\sigma_H^2\Big/2+\sqrt{\left(r-\sigma_H^2/2\right)^2+2\sigma_H^2(r+\lambda_H+\lambda_L)}}{\sigma_H^2},$$

$$k_2 = \frac{r-\sigma_H^2\Big/2-\sqrt{\left(r-\sigma_H^2/2\right)^2+2\sigma_H^2(r+\lambda_H+\lambda_L)}}{\sigma_H^2}.$$

Here, $k_1 > 1$ and $k_2 < 0$.

By an argument similar to the proof of Theorem 3.7, we can show that

$$c_1 = \int_{\underline{S}_L}^{\infty}\frac{L(t)}{(k_2-k_1)\sigma_H^2 t^{k_1-1}}dt \geq 0,$$

and

$$c_2 = l\underline{S}_L^{-k_2} - \underline{S}_L^{k_1-k_2}\int_{\underline{S}_L}^{\infty}\frac{L(t)}{(k_2-k_1)\sigma_H^2 t^{k_1-1}}.$$

Plugging these into (6.11), we know that h must be of the form

$$h(S) = \left[S^{k_1} - \left(\frac{S}{\underline{S}_L}\right)^{k_2}\underline{S}_L^{k_1}\right]\int_S^{\infty}\frac{L(t)}{(k_2-k_1)\sigma_H^2 t^{k_1-1}}dt$$

$$+\, l\left(\frac{S}{\underline{S}_L}\right)^{k_2} + \left(\frac{S}{\underline{S}_L}\right)^{k_2}\underline{S}_L\int_{\underline{S}_L}^{S}\frac{L(t)}{(k_2-k_1)\sigma_H^2}\left[\left(\frac{\underline{S}_L}{t}\right)^{k_2-1} - \left(\frac{\underline{S}_L}{t}\right)^{k_1-1}\right]dt.$$

Since $S^{k_1} - \left(\frac{S}{\underline{S}_L}\right)^{k_2}\underline{S}_L^{k_1} \geq 0$ for $S \geq \underline{S}_L$ and $\left(\frac{\underline{S}_L}{t}\right)^{k_2-1} \geq \left(\frac{\underline{S}_L}{t}\right)^{k_1-1}$ for all $t \geq \underline{S}_L$, $h(S) \geq 0$ for all $S > \underline{S}_L$. Strict inequality for $S > \underline{S}_L$ also follows from the above equation.

For $S \leq \underline{S}_L$, $P_L(S) = K - S \leq P_H(S)$ by convexity of $P_H(S)$ and the boundary condition $P_{H,S}\left(\underline{S}_H\right) = -1$. The strict inequality for $\underline{S}_H < S \leq \underline{S}_L$ follows from equation (3.5) and Corollary 3.6.

The proof is complete. ♠

Proof of Proposition 4.1. The proof is a slight modification of the proofs of Lemmas 3.3 and 3.4. ♠

Proof of Theorem 4.2. The proof proceeds in the same way as that of 3.7 and 3.8. ♠

Proof of Theorem 4.3. (i) Equations in (4.1) are derived in the next in this appendix. Since $P_m^{k-1}(t)$ is bounded and $n_2 < n_1 < 0 < 1 < n_4 < n_3$, from (6.16) we can show that

$$\lim_{S\to\infty}|S^{n_h}u_h^k(S)| < \infty \quad \text{for} \quad h = 1, 2,$$

and

$$\lim_{S\to\infty} |u_h^k(S)| < \infty \quad \text{for} \quad h = 3, 4.$$

Therefore, (4.2) follows immediately.

(ii) This result follows from a standard argument for a second-order ordinary differential equations.
♠

Proof of Theorem 4.4.

Use mathematical induction on k. Theorem 4.2 establishes the proof for $k = 1$. The rest of the proof is essentially the same as the proofs of Theorem 3.7 and Theorem 3.8. ♠

Derivation of Equation (4.1). Note that the homogeneous solution of the system of ordinary differential equations is of the following form with four undetermined constants C_h^k's:

$$\begin{aligned}\varphi_i^k(S) &= C_1^k S^{n_1} + C_2^k S^{n_2} + C_3^k S^{n_3} + C_4^k S^{n_4}, \quad \varphi_j^k(S)\\ &= C_1^k \xi_1 S^{n_1} + C_2^k \xi_2 S^{n_2} + C_3^k \xi_3 S^{n_3} + C_4^k \xi_4 S^{n_4}.\end{aligned}$$

In order to find the particular solution, we use the method of variation of parameters. We derive the matrix equation

$$X'(S) = P(S)X(S) + G(S), \tag{6.12}$$

where, for particular solutions ϕ_i^k and ϕ_j^k,

$$X(z) = \left(\phi_i^k(z),\ \phi_j^k(z),\ \phi_i^{k\prime}(z),\ \phi_j^{k\prime}(z)\right)^\top, \quad G(z) = \left(0,\ 0,\ -\frac{2\beta P_i^{k-1}(z)}{\sigma_i^2 z^2},\ -\frac{2\beta P_j^{k-1}(z)}{\sigma_j^2 z^2}\right)^\top$$

and

$$P(z) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \dfrac{2(r+\lambda_i+\beta)}{\sigma_i^2 z^2} & \dfrac{-2\lambda_i}{\sigma_i^2 z^2} & \dfrac{-2r}{\sigma_i^2 z} & 0 \\ \dfrac{2(r+\lambda_j+\beta)}{\sigma_j^2 z^2} & \dfrac{-2\lambda_j}{\sigma_j^2 z^2} & 0 & \dfrac{-2r}{\sigma_j^2 z} \end{pmatrix}.$$

Also set the *fundamental solution* Φ to be

$$\Phi(z) = \begin{pmatrix} z^{n_1} & z^{n_2} & z^{n_3} & z^{n_4} \\ \xi_1 z^{n_1} & \xi_2 z^{n_2} & \xi_3 z^{n_3} & \xi_4 z^{n_4} \\ n_1 z^{n_1-1} & n_2 z^{n_2-1} & n_3 z^{n_3-1} & n_4 z^{n_4-1} \\ \xi_1 n_1 z^{n_1-1} & \xi_2 n_2 z^{n_2-1} & \xi_3 n_3 z^{n_3-1} & \xi_4 n_4 z^{n_4-1} \end{pmatrix}. \tag{6.13}$$

Note that the fundamental solution satisfies the following equation

$$\Phi'(S) = P(S)\Phi(S). \tag{6.14}$$

Also note that $\Phi(z)$ is invertible for all $z > 0$ if and only if the determinant of $\Phi(1)$ is not equal to zero.

The method of variation of parameters is started by letting

$$X(S) \equiv \Phi(S)U(S) \equiv \Phi(S)\left(u_1^k(S),\ u_2^k(S),\ u_3^k(S),\ u_4^k(S)\right)^\top \tag{6.15}$$

From (6.12) and (6.14), we obtain

$$\Phi(S)U'(S) = G(S),$$

or equivalently,

$$U(S) = \int_{\underline{S}_j^k}^{S} \Phi^{-1}(t)G(t)dt.$$

Thus the particular solution can be represented as

$$\begin{pmatrix} \phi_i^k(S) \\ \phi_j^k(S) \end{pmatrix} = \begin{pmatrix} \sum_{h=1}^{4} S^{n_h} \cdot u_h^k(S) \\ \sum_{h=1}^{4} \xi_h S^{n_h} \cdot u_h^k(S) \end{pmatrix}.$$

Consequently, the solution is of the form

$$\begin{pmatrix} P_i^k(S) \\ P_j^k(S) \end{pmatrix} = \begin{pmatrix} \varphi_i^k(S) \\ \varphi_j^k(S) \end{pmatrix} + \begin{pmatrix} \phi_i^k(S) \\ \phi_j^k(S) \end{pmatrix} = \begin{pmatrix} \sum_{h=1}^{4} \left(C_h^k + u_h^k(S)\right) S^{n_h} \\ \sum_{h=1}^{4} \xi_h \left(C_h^k + u_h^k(S)\right) S^{n_h} \end{pmatrix}.$$

After immediate calculation, we obtain

$$u_h^k(S) = \int_{\underline{S}_j^k}^{S} t^{-n_h-1}\left(A_h P_i^{k-1}(t) + B_h P_j^{k-1}(t)\right)dt, \tag{6.16}$$

for some constants A_h and B_h. ♠

Corresponding author
Bong-Gyu Jang can be contacted at: bonggyujang@postech.ac.kr