Letter to the editor: Comments on the paper "derivation of lump solutions to a variety of Boussinesq equations with distinct dimensions"

The recent paper (Wazwaz, 2022) and several others of the same author are devoted to study a variety of nonlinear equations that are called Boussinesq equations in distinct dimensions. The author considers those equations as non-trivial generalizations of the classical Boussinesq equation:

$$u_{tt} + u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} = 0, \tag{1}$$

where $u(t, x)$ is an unknown smooth function (the lower subscripts denote differentiation with respect to relevant variables in what follows).

A new integrable (1 + 1)-dimensional Boussinesq equation is suggested in the form (Wazwaz, 2022):

$$u_{tt} + u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \alpha u_{xt} = 0 \tag{2}$$

(hereafter, the parameters $\alpha, \beta, \ldots$ are nonzero constants). However, if one applies the well-known technique used for the reduction of PDEs to their canonical forms (this technique is described in each textbook devoted to linear and quasi-linear PDEs, see, for instance, the classical book (Courant and Hilbert, 1962), then the PDE:

$$u_{t*t*} + (1 - \alpha^2/4)u_{x*x*} - \beta(u^2)_{x*x*} - \gamma u_{x*x*x*x*} = 0 \tag{3}$$

is obtained by the transformation:

$$t^* = t, \;\; x^* = x - \frac{\alpha}{2}\;t. \tag{4}$$

Obviously, PDE (3) is nothing else but the Boussinesq equation (1) in new notations. The coefficient $(1 - \alpha^2/4)$ is reducible to 1 by the transformation $\tau = \sqrt{1 - \alpha^2/4}t^*$, $|\alpha| \neq 2$, while the second term simply vanishes in the case $|\alpha| = 2$.

A new (1 + 2)-dimensional Boussinesq equation is suggested in the form (Wazwaz, 2022):

$$u_{tt} + u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \frac{\alpha^2}{4}u_{yy} + \alpha u_{yt} = 0 \tag{5}$$

The canonical form of the above equation reads as follows:

$$u_{y*y*} + u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} = 0 \tag{6}$$

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and is obtainable by the transformation:

$$t^* = t - \frac{2}{\alpha}\, y, \qquad y^* = \frac{2}{\alpha}\, y. \tag{7}$$

Obviously, PDE (6) is again the Boussinesq equation (1) in new notations.

Finally, the so-called (1 + 3)-dimensional Boussinesq equation is proposed in the form (Wazwaz, 2022):

$$u_{tt} + u_{xx} - \beta(u^2)_{xx} - \gamma u_{xxxx} + \frac{\alpha^2}{4} u_{yy} + \alpha u_{yt} + \delta u_{xz} = 0. \tag{8}$$

The above PDE is reducible to the much simpler equation:

$$u_{y*y*} - \beta(u^2)_{x*x*} - \gamma u_{x*x*x*x*} + \delta u_{x*z} = 0 \tag{9}$$

by the transformation:

$$t^* = t - \frac{2}{\alpha} y, \qquad x^* = x - \frac{1}{\delta} z, \qquad y^* = \frac{2}{\alpha} y. \tag{10}$$

It is very difficult to imagine that PDE (9) is a (1 + 3)-dimensional Boussinesq equation if one compares this equation and PDE (1). On the other hand, one easily notes that PDE (9) coincides (up to notations and parameter signs) with the classical Kadomtsev–Petviashvili equation:

$$u_{yy} - \lambda\left(u_{xt} + \frac{3}{2}(u^2)_{xx} + \gamma u_{xxxx}\right) = 0 \tag{11}$$

So, PDE (8) is equivalent to the KP equation and cannot be called the (1 + 3)-dimensional Boussinesq equation.

Finally, it is a well-known fact that the Boussinesq equation and the KP equation are integrable. So, integrability and other properties of the three equations investigated in (Wazwaz, 2022) and several other papers are trivial consequences of the integrability of the classical equations (1) and (11).

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References

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