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Cauchy type nonlinear inverse problem in a two-layer area

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Abstract

Purpose – To reduce the heat load of a gas turbine blade, its surface is covered with an outer layer of ceramics with high thermal resistance. The purpose of this paper is the selection of ceramics with such a low heat conduction coefficient and thickness, so that the permissible metal temperature is not exceeded on the metal-ceramics interface due to the loss ofmechanical properties.

Design/methodology/approach – Therefore, for given temperature changes over time on the metal-ceramics interface, temperature changes over time on the inner side of the blade and the assumed initial temperature, the temperature change over time on the outer surface of the ceramics should be determined. The problem presented in this way is a Cauchy type problem. When analyzing the problem, it is taken into account that thermophysical properties of metal and ceramics may depend on temperature. Due to the thin layer of ceramics in relation to the wall thickness, the problem is considered in the area in the flat layer. Thus, a one-dimensional non-stationary heat flow is considered.

Findings – The range of stability of the Cauchy problem as a function of time step, thickness of ceramics and thermophysical properties of metal and ceramics are examined. The numerical computations also involved the influence of disturbances in the temperature on metal-ceramics interface on the solution to the inverse problem.

Practical implications – The computational model can be used to analyze the heat flow in gas turbine blades with thermal barrier.

Originality/value – A number of inverse problems of the type considered in the paper are presented in the literature. Inverse problems, especially those Cauchy-type, are ill-conditioned numerically, which means that a small change in the inputs may result in significant errors of the solution. In such a case, regularization of the

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inverse problem is needed. However, the Cauchy problem presented in the paper does not require regularization.

Keywords Turbine blade, Cauchy type problem, Energy balance equation, Inverse heat conduction problem, Two-layer area

Paper type Research paper

Nomenclature

$a^n_{i,i-1}, a^n_{i,i}, a^n_{i,i+1}$	= elements of matrix $[a^n]$; see equation (12), [–];
β^n_i	= elements of matrix $[\beta^n]$; see equation (12), [–];
c_m, c_c	= specific heat of metal and ceramics, respectively [J/kgK];
δ	= thickness of ceramics [mm];
g	= thickness of metal layer [mm];
d	= thickness of the wall ($d = g + \delta$)[mm];
γ	= element of matrix $[a]$; see equation (17);
λ_m, λ_c	= heat conduction coefficient of the metal and ceramics, respectively [W/mK];
Q_i	= temperature at the n^{th} moment at point x_i, [°C];
ρ_m, ρ_c	= density of metal and ceramics, respectively [kg/m^3];
ρ	= spectra radius of the stability matrix; equation (26);
σ_1	= maximal singular value of the stability matrix [-];
t	= time variable [s];
$T(x,t)$	= temperature [°C];
Δt	= time step [s];
u^n_k, w^n_k	= parameters defining $\tilde{w}^n_k$; see equation (20), [W/(m^{2}K)];
$\tilde{w}^n_k$	= element of matrix $[a]$; see equation (19), [W/(m^{2}K)];
x	= space variable [m];
Δx	= space step [m]; and
q_g	= heat flux on the metal-ceramics interface.

1. Introduction

To reduce the thermal load of a blade in a gas turbine, the blade's outer surface is coated with a layer of a ceramics of a high thermal resistance. The crucial problem in this area is such selection of the ceramics conductivity and thickness that the permissible temperature of the metal on the metal-ceramics interface is not exceeded. Otherwise, the metal may lose its mechanical properties. Hence, for the prescribed changes of temperature over time on the metal-ceramics interface, given temperature on the inner side of the blade and the initial temperature, the changes of temperature over time on the outer surface of the blade should be determined. This is the inverse problem of the Cauchy type.

The bibliography of inverse issues is very rich and covers various types of problems. The inverse problems are ill-posed, the solutions are not stable with respect to perturbation on the input data and the results are frequently not unique. These may include problems of identifying thermal conditions on a part of the boundary of the studied area, identifying the shape of the area, thermophysical coefficients, sources and more.

Many monographs and publications were devoted to the ill-posed and inverse problems and methods of searching approximated and stable solutions (Alifanov, 1994; Bakushinskii and Goncharsky, 1995; Engl *et al.*, 2000; Gockenbach, 2016; Kurpisz and Nowak, 1995; Ramm, 2004; Tikhonov and Arsenin, 1977) and other. Inverse problems of various types have been considered in many publications. In particular, these papers involved the Cauchy problem and the heat flow in a two-layer area. The method presented in Caillé *et al.* (2019)

refers to the three-dimensional Helmholtz equation in which the solution of the Helmholtz equation obtained from the solution of the Dirichlet problem with the values of the normal derivative on a part of the boundaries is reproduced. The issue considered in Marin (2010) concerns heat flow in a multilayer area with different thermophysical properties of the partition material for a non-stationary one-dimensional case. The paper (Liu and Wei, 2011) concerns a solution of a non-stationary linear direct problem in a multilayer area using the Fourier transform. Each layer has different thermophysical properties. In Simões *et al.* (2012), a solution to the non-stationary problem of one-dimensional heat conduction equation in a two-layer area with different thermophysical properties of each (partition) materials was considered. The problem is formulated as the Cauchy problem with zero initial temperature. The inverse problem was solved in the frequency domain using the modified Tikhonov regularization method. The solution of the Cauchy problem in multilayered domain was implemented in the frequency domain by applying the Fourier transformation method (Xiong and Hon, 2013). The authors stated that the modified Tikhonov regularization (implemented in the frequency domain) is more efficient as the classical approach to this method. Yang *et al.* (2019) used the modified Tikhonov regularization and the truncation method for solving the Cauchy problem of the Helmholtz equation. A Cauchy problem on the semiline for a non-linear diffusion equation is considered in De Lillo *et al.* (2006), with a boundary condition corresponding to a prescribed thermal conductivity at the origin. The problem is mapped into a moving boundary problem for the linear heat equation with a Robin-type boundary condition. Such a problem is then reduced to a linear integral Volterra equation of II type, which admits a unique solution. In Marin and Lesnic (2005), the application of the method of fundamental solutions to the Cauchy problem associated with two-dimensional Helmholtz-type equations is investigated. The resulting system of linear algebraic equations is ill-conditioned and therefore its solution is regularized by using the first-order Tikhonov functional, while the choice of the regularization parameter is based on the L-curve method. In Haò (1995), a mathematical consideration concerning the Cauchy problem is presented. Non-characteristic Cauchy problems for parabolic equations are frequently encountered in many areas of the heat transfer. These problems are well-known to be severely ill-posed. In this paper, a solvability criterion for a class of such problems is established.

In Liu and Wei (2011), the authors transformed the original ill-posed problem into a well-posed problem. They implemented method of lines to reconstruct a stable approximation of the moving boundary. An analytical formulation of the temperature distribution in multi-layered and multi-dimensional bodies was performed in Haji-Sheikh *et al.* (2003). The authors performed a numerical simulation of steady-state heat conduction for two-layered bodies in steady state – they indicate that their steady-state simulation has a high level of accuracy if each layer is homogeneous or orthotropic.

Solutions to inverse problems were applied for analyzing the heat flow in gas-turbine blades (Frąckowiak *et al.*, 2017, 2019b) and other crucial thermal and flowing inverse problems (Grysa *et al.*, 2012; Joachimiak *et al.*, 2019b; Joachimiak and Krzyślak, 2019). The authors analyzed also many other inverse problem available in the literature (Frąckowiak *et al.*, 2019a; Grysa *et al.*, 2014, 2018; Joachimiak and Ciałkowski, 2018; Maciag and Jehad Al-Khatib, 2000; Maciąg and Grysa, 2016).

The presented paper addresses an inverse problem in a two-layered domain. A layer of the ceramics is thin comparing to the thickness of metal layer. Also, the thickness of the two layers together is relatively thin, so it is assumed that the problem is examined in the domain of a flat layer. Therefore, a one-dimensional unsteady flow is considered. The problem is non-linear because thermophysical properties of both layers depend on temperature. Previously, a one-dimensional transient Cauchy problem with the thermophysical properties assumed to be temperature independent has been solved in a two-layer domain (Ciałkowski *et al.*, 2020).

Discretization with respect to time and space is applied. The temperature is set on the metal-ceramics interface. The purpose of the calculation is to determine the outer surface temperature. Then calculations are carried out under the conditions of equal heat flux and temperature on the interface. The system of equations was supplemented by the condition of energy conservation in integral form in the area of ceramics to stabilize an unknown temperature on its outer surface. This leads to the inverse Cauchy problem, but the condition of energy conservation is ensured the stability of the inverse solution. Such problem is ill-posed in the Hadamard sense (Alifanov, 1994; Hadamard, 1902; Tikhonov and Arsenin, 1977) and generally needs a regularization (Frąckowiak and Ciałkowski, 2018; Joachimiak, 2020; Joachimiak *et al.*, 2019a, 2016). However, the adopted method of calculation makes the problem under consideration possible to be solved without regularization. The range of stability of the Cauchy problem in the function of time step, of the ceramics thickness and of thermophysical properties of the metal and the ceramics is investigated.

2. Basic equations

Figure 1 shows a two-layer computing domain. In domain $\langle 0, g \rangle$, the heat conducting material is metal with thermophysical coefficients λ_m, ρ_m, c_m, and in the domain $x \in \langle g, g + \delta \rangle$, the conductive material is ceramics with thermophysical properties λ_c, ρ_c, c_c. *In a particular case, the entire area can be filled with metal.*

In the two-layer area under consideration, the equations describing the heat flow are as follows:

- metal area (index *m* refers to the metal)

$$\rho_m c_m \frac{\partial T_m}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_m \frac{\partial T_m}{\partial \lambda x}\right), x \in \langle 0, g \rangle, t > 0 \tag{1}$$

- ceramics area (index *c* refers to the ceramics)

$$\rho_c c_c \frac{\partial T_c}{\partial t} = \frac{\partial}{\partial x}\left(\lambda_c \frac{\partial T_c}{\partial x}\right), x \in \langle g, g + \delta \rangle, t > 0 \tag{2}$$

with the following conditions

- initial condition, for $t = 0$

$$T(x, 0) = f(x), x \in \langle 0, g + \delta \rangle \tag{3}$$

- boundary condition on the surface $x = 0$

$$T(0, t) = T_0(t) \tag{4}$$

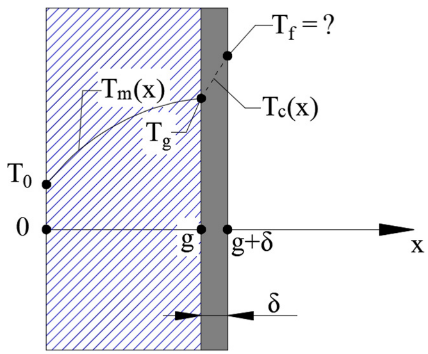

Figure 1.
Computing domain

- condition of equality of temperature and heat flux (energy) streams on both sides of the metal-ceramics interface, $x = g$

$$T_m(g,t) = T_c(g,t) = T_g(t) \tag{5}$$

$$q_g = \lambda_m \frac{\partial T_m(g,t)}{\partial x} = \lambda_c \frac{\partial T_c(g,t)}{\partial x} \tag{6}$$

In fact, equation (2) with conditions (3), (5) and (6) is a classic Cauchy problem particularly sensitive to data inaccuracies. In general, the Cauchy problem requires regularization, which results from the instability of the solution. This instability is the result of numerical errors and hence failure to fulfil energy conservation in the area of ceramics. Solutions of equations (1) and (2) with conditions (3)–(6) will be made for the temperature dependent thermophysical parameters.

The essence of the problem consists in finding how the temperature of the outer surface of the ceramics changes with time during heating up, provided the temperature T_g on the metal-ceramics interface is known from measurement or by assumption. The reason for the latter is that T_g may not exceed the level resulting in the loss of mechanical properties of the metal. Consequently, it determines admissible value of the ceramics surface temperature T_f. The heat flux on the metal-ceramics interface, q_g (6), is unknown, and it determines a thermal transmission condition.

3. Linearization of the conduction equation

Equations (1) and (2) without reference to the type of thermally conductive material (indexes are omitted) can be written as follows:

$$\rho(T)c(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) \tag{7}$$

The solution will be sought by the method of subsequent iterations in which the coefficients *ρ(T), c(T), λ(T)* will be determined for the temperature from the previous iteration step. Therefore, equation (7) takes the form:

$$\rho(T^n)c(T^n)\frac{\partial T^{n+1}}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T^n)\frac{\partial T^{n+1}}{\partial x}\right), n = 1,2,\ldots \tag{8}$$

and the iterative process ends when $\|T^{n+1} - T^n\| < \varepsilon$.

We will look for the solution of equation (8) in a discrete form on the grid (Figure 2). This equation will be transformed to a discrete form. Discretization of the derivative with respect to time (back differential quotient) and space (central differential quotient) leads to the equation:

$$\rho(T_i^n)c(T_i^n)\frac{T_i^{n+1} - Q_i}{\Delta t} = \frac{\lambda\left(\frac{T_{i+1}^n + T_i^n}{2}\right)\cdot\frac{T_{i+1}^{n+1} - T_i^{n+1}}{x_{i+1} - x_i} - \lambda\left(\frac{T_i^n + T_{i-1}^n}{2}\right)\cdot\frac{T_i^{n+1} - T_{i-1}^{n-1}}{x_i - x_{i-1}}}{\frac{x_{i+1}+x_i}{2} - \frac{x_i + x_{i-1}}{2}} \tag{9}$$

$$\lim_{n\to\infty} T_i^{n+1} = T(x_i, t - \Delta t) = Q_i\,,\ i = 1,2,\ldots \tag{10}$$

Substitute

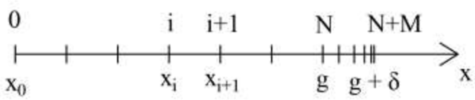

Figure 2.
Discrete grid of a two-layer area

$$h_i = x_{i+1} - x_i\,,\ h_{i1} = x_i - x_{i-1},$$

$$h_{is} = \frac{x_{i+1} + x_i}{2} - \frac{x_i + x_{i-1}}{2} = \frac{x_{i+1} - x_i}{2} + \frac{x_i - x_{i-1}}{2} = \frac{h_i + h_{i1}}{2}$$

$$\lambda_i = \lambda\left(\frac{T_{i+1}^n + T_i^n}{2}\right),\ \ \lambda_{i1} = \lambda\left(\frac{T_i^n + T_{i-1}^n}{2}\right),\ \lambda_{is} = \frac{\lambda_i + \lambda_{i1}}{2}, \rho_i = \rho\left(T_i^n\right), c_i = c\left(T_i^n\right)$$

$$\frac{\rho_i c_i}{\lambda_{is}\Delta t}\left(T_i^{n+1} - Q_i\right) = \frac{1}{h_{is}}\left[\frac{\lambda_i}{\lambda_{is} h_i}\left(T_{i+1}^{n+1} - T_i^{n+1}\right) - \frac{\lambda_{i1}}{\lambda_{is} h_{i1}}\left(T_i^{n+1} - T_{i-1}^{n+1}\right)\right]$$

Then the formula (9) takes the form:

$$\frac{\lambda_{i1} h_{is}}{\lambda_{is} h_{i1}} T_{i-1}^{n+1} - \left(\frac{\lambda_i h_{is}}{\lambda_{is} h_i} + \frac{\lambda_{i1} h_{is}}{\lambda_{is} h_{i1}} + \frac{\rho_i c_i h_{is}^2}{\lambda_{is}\Delta t}\right) T_i^{n+1} + \frac{\lambda_i h_{is}}{\lambda_{is} h_i} T_{i+1}^{n+1} = -\frac{\rho_i c_i h_{is}^2}{\lambda_{is}\Delta t} Q_i \tag{11}$$

Accepting designation

$$a_{i,i-1}^n = \frac{\lambda_{i1} h_{is}}{\lambda_{is} h_{i1}}, a_{ii}^n = \frac{\lambda_i h_{is}}{\lambda_{is} h_i} + \frac{\lambda_{i1} h_{is}}{\lambda_{is} h_{i1}} + \frac{\rho_i c_i h_{is}^2}{\lambda_{is}\Delta t}, a_{i,i+1}^n = \frac{\lambda_i h_{is}}{\lambda_{is} h_i}, \beta_i^n = \frac{\rho_i c_i h_{is}^2}{\lambda_{is}\Delta t} \tag{12}$$

equation (11) can be rewritten as follows:

$$a_{i,i-1}^n T_{i-1}^{n+1} + a_{ii}^n T_i^{n+1} + a_{i,i+1}^n T_{i+1}^{n+1} = -\beta_i^n Q_i,\ i = 1, 2, \ldots \tag{13}$$

or in the matrix form

$$\begin{bmatrix} a_{10}^n a_{11}^n & a_{12}^n & & & & \\ a_{21}^n & a_{22}^n & a_{23}^n & & & \\ & a_{32}^n & a_{33}^n & a_{34}^n & & \\ & \ddots & \ddots & \ddots & & \\ & & & a_{N-1,N-2}^n & a_{N-1,N-1}^n & a_{N-1,N}^n \end{bmatrix} \begin{Bmatrix} T_0 \\ T_1 \\ \vdots \\ T_{N-1} \\ T_N \end{Bmatrix} = \begin{bmatrix} -\beta_1^n & & & & \\ & -\beta_2^n & & & \\ & & -\beta_3^n & & \\ & & & \ddots & \\ & & & & -\beta_{N-1}^n \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{N-2} \\ Q_{N-1} \end{Bmatrix} \tag{14}$$

For the ceramics layer, the set of mesh internal points is as follows $x_{N+1}, x_{N+2}, \ldots, x_{N+M-1}$. The conduction equation is met inside the test area. However, in view of the adopted discretization technique, the point on the metal-ceramics interface is omitted. Thus, the matrix corresponding to the equation of conduction in the two-layer area has the following form:

$$\begin{array}{c} \\ \\ i = N-1 \\ \\ i = N+1 \\ \\ \\ \end{array} \begin{bmatrix} a_{10}^n & a_{11}^n & a_{12}^n & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & \vdots & 0 & \text{metal} & 0 & 0 \\ 0 & & & a_{N-1,N-2}^n\, a_{N-1,N-1}^n\, a_{N-1,N}^n & 0 & 0 & & & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & a_{N+1,N}^n & \vdots & a_{N+1,N+1}^n & a_{N+1,N+2}^n & 0 & 0 \\ 0 & \text{ceramics} & 0 & 0 & \vdots & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \vdots & 0 & a_{N+M-1,N+M-2}^n\, a_{N+M-1,N+M-1}^n\, a_{N+M-1,N+M}^n & & \end{bmatrix} \begin{Bmatrix} T_0^{n+1} \\ T_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T_{N+M}^{n+1} \end{Bmatrix} =$$

$$= \begin{bmatrix} -\beta_1^n & & & \vdots & 0 & 0 & 0 \\ & \ddots & & \vdots & & & 0 \\ 0 & 0 & -\beta_{N-1}^n & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & -\beta_{N+1}^n & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & -\beta_{N+M}^n \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ \vdots \\ \vdots \\ Q_{N+M-2} \\ Q_{N+M-1} \end{Bmatrix}$$

or in a compact form

$$[a^n]\{T^{n+1}\} = [\beta^n]\{Q\},\ dim[a^n] = (N-1+M-1)\times(N+M) \tag{15}$$

The overall dimension of the matrix [a^n] is $(N+M-2)\times(N+M)$. The remaining elements of the matrix will be supplemented with the boundary condition (4) and the assumed temperature on the interface of the layers metal–ceramics, i.e. with the condition (5).

In the interval $x\epsilon\langle g,\ g+\delta\rangle$, the Cauchy problem for the equation (2) is solved. Therefore, it is necessary to know the temperature and heat flux at $x=g=x_N$. The condition of the heat flux equality on both sides of the interface is expressed as follows:

$$\lambda_m(T_N^n)\frac{dT_m^n}{dx}\bigg|_{x=x_N} = \lambda_c(T_N^n)\frac{dT_c^n}{dx}\bigg|_{x=x_N}$$

where index m refers to the metal layer and index c to the ceramics layer. In the discrete form, the condition reads as follows:

$$\lambda_m(T_N^n)\frac{T_N^n - T_{N-1}^n}{x_N - x_{N-1}} = \lambda_c(T_N^n)\frac{T_{N+1}^n - T_N^n}{x_{N+1} - x_N} \tag{16}$$

Hence, the heat fluxes compliance condition is finally expressed by the following formula:

$$T_{N-1}^n - (1+\gamma)T_N^n + \gamma T_{N+1}^n = 0,\ \ \gamma = \frac{\lambda_c(T_N^n)}{\lambda_m(T_N^n)}\cdot\frac{x_N - x_{N-1}}{x_{N+1} - x_N} \tag{17}$$

Due to the discrete form of the equation describing heat flow, it is necessary to take into account the conditions on the boundaries of the ceramics layer, which will allow to fulfil energy balance equation in the closed interval $\langle g,\ g+\delta\rangle$. This can be obtained by requiring the integral equation to be satisfied. In addition, it imposes a condition on the unknown temperature T_{N+M} and makes the inverse problem stable. Integrating equation (2) in the area of ceramics $\langle g,\ g+\delta\rangle$ we get the following:

$$\int_{g=x_N}^{g+\delta=x_{N+M}} \frac{\rho(T^n)c(T^n)}{\Delta t}\left(T^{n+1} - Q\right)dx = \lambda(T^n)\frac{dT^{n+1}}{dx}\Big|_{x=x_{N+M}} - \lambda(T^n)\frac{dT^{n+1}}{dx}\Big|_{x=x_N} \tag{18}$$

Numerical integration method with parameter Θ (for the trapezoidal method $\Theta = 0.5$) used to calculate the integral to the left side of equation (18) (denoted I) leads to the following results:

$$I \cong \sum_{k=N}^{k=N+M-1} \left[\frac{\rho(T_k^n)c(T_k^n)}{\Delta t}(T_k^{n+1} - Q_k)\Theta + \frac{\rho(T_{k+1}^n)c(T_{k+1}^n)}{\Delta t}(T_{k+1}^{n+1} - Q_{k+1})(1-\Theta) \right](x_{k+1} - x_k) =$$

$$= \frac{\rho(T_N^n)c(T_N^n)}{\Delta t}(T_N^{n+1} - Q_N)\Theta(x_{N+1} - x_N) +$$

$$+ \sum_{k=N+1}^{k=N+M-1} \left[\frac{\rho(T_k^n)c(T_k^n)}{\Delta t}(T_k^{n+1} - Q_k)\Theta(x_{k+1} - x_k) + \frac{\rho(T_k^n)c(T_k^n)}{\Delta t}(T_k^{n+1} - Q_k)(1-\Theta)(x_k - x_{k-1}) \right]$$

$$+ \frac{\rho(T_{N+M}^n)c(T_{N+M}^n)}{\Delta t}(T_{N+M}^{n+1} - Q_{N+M})(1-\Theta)(x_{N+M} - x_{N+M-1}) =$$

$$= \frac{\rho(T_N^n)c(T_N^n)}{\Delta t}(T_N^{n+1} - Q_N)\Theta(x_{N+1} - x_N) +$$

$$+ \sum_{k=N+1}^{k=N+M-1} \left[\frac{\rho(T_k^n)c(T_k^n)}{\Delta t}(T_k^{n+1} - Q_k)[\Theta(x_{k+1} - x_k) + (1-\Theta)(x_k - x_{k-1})] \right] +$$

$$+ \frac{\rho(T_{N+M}^n)c(T_{N+M}^n)}{\Delta t}(T_{N+M}^{n+1} - Q_{N+M})(1-\Theta)(x_{N+M} - x_{N+M-1})$$

Finally

$$I = \sum_{k=N}^{k=N+M} w_k^n T_k^{n+1} - \sum_{k=N}^{k=N+M} w_k^n Q_k \tag{19}$$

Here

$$w_k^n = \begin{cases} \frac{\rho(T_N^n)c(T_N^n)}{\Delta t}\Theta(x_{N+1} - x_N) & \text{for } k = N, \\ \frac{\rho(T_k^n)c(T_k^n)}{\Delta t}\left[\Theta(x_{k+1} - x_k) + (1-\Theta)(x_k - x_{k-1})\right] & \text{for } k = N+1, \ldots, N+M-1, \\ \frac{\rho(T_{N+M}^n)c(T_{N+M}^n)}{\Delta t}(1-\Theta)(x_{N+M} - x_{N+M-1}) & \text{for } k = N+M. \end{cases}$$

For constant values of density ρ, specific heat c and temperatures T and Q, the value of integral I is as follow:

$$I = \frac{\rho c}{\Delta t}(T - Q)\delta$$

The right side of equation (18) can be transformed as follows:

$$\lambda(T^n)\frac{dT^{n+1}}{dx}|_{x_{N+M}} - \lambda(T^n)\frac{dT^{n+1}}{dx}|_{x_N} \cong \lambda(T_{N+M}^n)\frac{T_{N+M}^{n+1} - T_{N+M-1}^{n+1}}{x_{N+M} - x_{N+M-1}} - \lambda(T_N^n)\frac{T_{N+1}^{n+1} - T_N^{n+1}}{x_{N+1} - x_N} =$$

$$= \left(T_N^{n+1} - T_{N+1}^{n+1}\right)\frac{\lambda(T_N^n)}{x_{N+1} - x_N} - \left(T_{N+M-1}^{n+1} - T_{N+M}^{n+1}\right)\frac{\lambda(T_{N+M}^n)}{x_{N+M} - x_{N+M-1}} = \sum_{k=N}^{k=N+M} u_k^n T_k^{n+1}$$

where

$$\{u^n\} = \left\{ \frac{\lambda(T_N^n)}{x_{N+1} - x_N}, \frac{-\lambda(T_N^n)}{x_{N+1} - x_N}, 0, \ldots, 0, \frac{-\lambda(T_{N+M}^n)}{x_{N+M} - x_{N+M-1}}, \frac{\lambda(T_{N+M}^n)}{x_{N+M} - x_{N+M-1}} \right\}^T$$

Hence, the energy balance equation in the ceramics layer takes the following form:

$$\sum_{k=N}^{k=N+M} w_k^n T_k^{n+1} - \sum_{k=N}^{k=N+M} w_k^n Q_k = \sum_{k=N}^{k=N+M} u_k^n T_k^{n+1}$$

or

$$\sum_{k=N}^{k=N+M} \tilde{w}_k^n T_k^{n+1} = \sum_{k=N}^{k=N+M} w_k^n Q_k \tag{20}$$

with $\tilde{w}_k^n = w_k^n - u_k^n$.

Equations (17) and (20) complement the system of equations (15) written in a matrix form in a two-layer area. So finally, the matrix of equations has the following form:

$$\begin{bmatrix} a_{10}^n & a_{11}^n & a_{12}^n & 0 & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & \ddots & & & 0 & \vdots & 0 & & & \\ 0 & & a_{N-1,N-2}^n & a_{N-1,N-1}^n & a_{N-1,N}^n & \vdots & 0 & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & a_{N+1,N}^n & \vdots & a_{N+1,N+1}^n & a_{N+1,N+2}^n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \vdots & 0 & a_{N+M-2}^n & a_{N+M-1}^n & a_{N+M}^n \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & 1-\gamma & \vdots & \gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \tilde{w}_N^n & \vdots & \tilde{w}_{N+1}^n & 0 & 0 & \tilde{w}_{N+M}^n \end{bmatrix} \begin{Bmatrix} T_0^{n+1} \\ T_1^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T_{N+M}^{n+1} \end{Bmatrix} =$$

$$= \begin{bmatrix} -\beta_1^n & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 0 & -\beta_{N-1}^n & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & \vdots & -\beta_{N+1}^n & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & & & \vdots & 0 & \vdots & & & -\beta_{N+M-1}^n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & & 0 & \vdots & w_N^n & \vdots & w_{N+1}^n & 0 & 0 & w_{N+M}^n \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ Q_{N-1} \\ Q_N \\ Q_{N+1} \\ \vdots \\ \vdots \\ \vdots \\ Q_{N+M} \end{Bmatrix} \tag{21}$$

or

$$[a]\{T\} = [\beta]\{Q\} \tag{22}$$

$$dim[a] = (N+M) \times (N+M)\,,\; dim[\beta] = (N+M) \times (N+M)$$

Because the temperature $T = T_0$ is set at $x_0 = 0$ and the temperature $T = T_g$ is at $x_N = g$, so the equation (21) can be written as follows:

$$T_N = T_g$$

$$\begin{bmatrix} a_{11}^n & a_{12}^n & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 \\ 0 & & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & & \vdots & 0 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 1 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 1 & \vdots & 0 & \vdots & \gamma & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & \vdots & \tilde{w}_{N+1}^n & \cdots & \tilde{w}_{N+M}^n \end{bmatrix} \begin{bmatrix} T_1^{n+1} \\ T_2^{n+1} \\ \vdots \\ \vdots \\ T_N^{n+1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ T_{N+M}^{n+1} \end{bmatrix} = \begin{bmatrix} -a_{10}^n \\ 0 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ 0 \end{bmatrix} T_0 + \begin{bmatrix} -a_{1N}^n \\ -a_{2N}^n \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ -a_{N+M,N}^n \end{bmatrix} T_g =$$

$$= \begin{bmatrix} -\beta_1^n & 0 & 0 & \vdots & 0 & \vdots & & & & 0 \\ 0 & \ddots & 0 & \vdots & 0 & \vdots & 0 & & & 0 \\ 0 & 0 & -\beta_{N-1}^n & \vdots & 0 & \vdots & & & & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & \vdots & -\beta_{N+1}^n & 0 & & 0 \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & \ddots & 0 & 0 \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & -\beta_{N+M-1}^n & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & 0 & \vdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \vdots & w_N^n & \vdots & w_{N+1}^n & 0 & 0 & w_{N+M}^n \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ \vdots \\ \vdots \\ Q_{N-1} \\ Q_N \\ Q_{N+1} \\ \vdots \\ \vdots \\ \vdots \\ Q_{N+M} \end{Bmatrix} \tag{23}$$

or in a simpler form:

$$\left[a_{new}^n\right]\left\{T^{n+1}\right\} = \left\{b_0^n\right\}T_0 + \left\{b_g^n\right\}T_g + \left[\beta_{new}^n\right]\{Q\} \tag{24}$$

$$dim\left[a_{new}^n\right] = dim\left[\beta_{new}^n\right],\; dim\left\{T^{n+1}\right\} = dim\left\{b_0^n\right\} = dim\left\{b_g^n\right\} = N+M.$$

Hence, the solution is as follows:

$$\{T^{n+1}\} = [a^n_{new}]^{-1}\Big\{[\beta^n_{new}]\{Q\} + \{b^n_0\}T_0 + \{b^n_g\}T_g\Big\} \tag{25}$$

For $n \to \infty$, the stability matrix has the following form:

$$[a_{stabil}] = \lim_{n\to\infty} [a^n_{new}]^{-1}[\beta^n_{new}] \tag{26}$$

For the non-linear problem, the stability matrix $[a_{stabil}]$ has a different form for each subsequent time step.

5. Numerical calculations

Numerical calculations were carried out for the following metal and ceramics (zirconium oxide) parameters. For steel, constant values of the heat conduction coefficient $\lambda = 30$ [W/mK], density $\rho = 7850$ [kg/m^3] and specific heat $c = 450$ [J/kgK] were adopted, while for ceramics variable parameters λ, ρ and c accepted. Their variability with temperature is shown in Figures 3, 4 and 5.

A Cauchy problem is numerically ill-conditioned, which means that a small change in the inputs leads to a large change in the final results. Therefore, the response of ceramics surface temperature T_f to disturbances of temperature T_g on the metal-ceramics interface has been analyzed. A sample distribution of temperature T_g, disturbed and undisturbed, is shown in Figure 6. The errors reached $\pm 1\%$ of undisturbed values. The temperature T_g impacts the heat flux on the metal-ceramics interface q_g, which is presented in Figure 7.

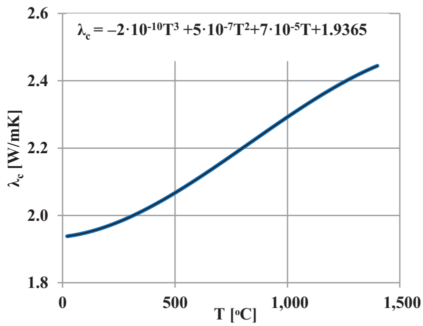

Figure 3. Heat conduction coefficient λ_c for zirkonium oxyde as a function of temperature

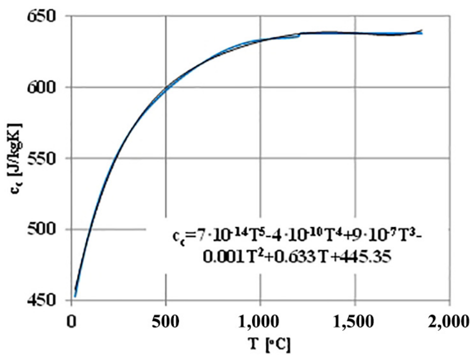

Figure 4. Specific heat c_c for zirkonium oxyde as a function of temperature

Propagation of errors in the considered Cauchy problem has been analyzed taking into account the influence of disturbances to thermophysical properties (λ, ρ, c) of the material on temperature T_f. Figures 8 and 9 show disturbed heat conduction coefficient of the metal (λ_m) and ceramics $(\lambda)_c$.

The iterative process ends when $\|T^{n+1} - T^n\| < \varepsilon$, $\varepsilon = 0.0000001$. This accuracy of calculations is achieved after five iterations. In a thin layer of ceramics, the temperature distribution is practically linear, so three internal nodes in this area are sufficient. In calculations, their number for a ceramics layer equal to 0.5 mm is 5. For the metal layer, the number of internal nodes is 16. The mesh, however, is constructed so that the lengths of the intervals adjacent to the interface x = g are the same.

Condition number of matrix [a_{new}] (equation (24) for the ceramics layer thickness δ of $0.1 \div 0.3$ mm decreases with time (Figure 10), and its values are not significantly affected by disturbances to temperature T_g.

For the assumed values of thermophysical coefficients and given conditions at $x = 0$, $x = g$ (metal-ceramics interface), the spectral radius of the stability matrix (26) was analyzed for a constant time step $\Delta t = 1$ [s] and different thickness of the ceramics layer. Due to the non-linearity of thermophysical parameters, the spectral radius of the stability matrix has a different value for the next time. The changes over time in the spectral radius of the stability matrix are shown in Figure 11. The values of the spectral radius obtained for disturbed and

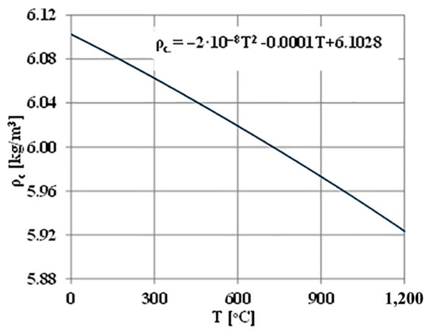

Figure 5.
Density ρ_c for zirkonium oxyde as a function of temperature

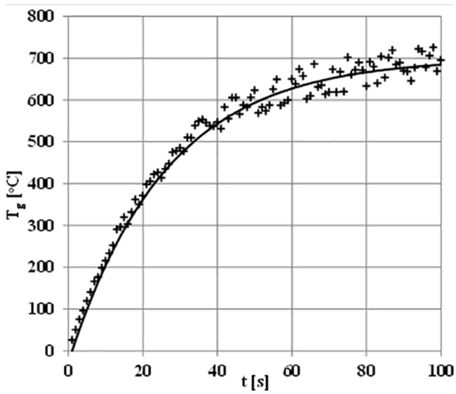

Figure 6.
A sample distribution of temperature T_g on the metal-ceramics interface: undisturbed (solid line) and with temperature disturbances $T_g \pm 1\%$ (plus signs)

undisturbed values of temperature T_g do not considerably differ from one another (Figure 11), which implies stability of the achieved solution. The magnitudes of the condition number and spectral radius (as shown in Figure 10 and Figure 11, respectively) indicate regularization properties of the applied equation (18).

For the temperature $T = T_0 + T_g\,(1 - e^{0.01t})$ set at point $x = g$ (bottom line in Figure 12), the temperature changes over time on the outer surface of the ceramics are presented as a function of the ceramics layer thickness for a metal layer thickness $\delta_m = g = 5$ mm. For ceramics thickness $\delta = 0.5$ mm, the temperature difference between the outer surface of the ceramics and the ceramics-metal interface is about 900 [°C] (see Figure 12, bottom and top line). Results are pesented for disturbed and undisturbed values of temperature T_g on the metal-ceramics interface.

The temperature distribution in the ceramics layer after the first second and after 200 s is linear, as shown in Figure 13.

A very important issue is the monitoring of the heat load in a homogeneous material (for example in the body of a steam or gas turbine) based on the temperature measurement at the internal point. The question then arises as to how far (how deep) the thermocouple can be placed so that the solution of the inverse problem is stable. This issue was solved for G20Mo5 steel characterized by variable thermophysical parameters as functions of temperature, which is shown in Figures 14, 15 and 16.

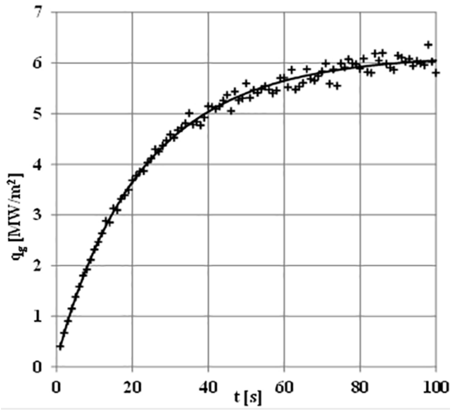

Figure 7.
A sample distribution of the heat flux on the metal-ceramics interface: undisturbed (solid line) and with disturbances to temperature $T_g \pm 1\%$ and heat conduction coefficient $\pm 2.5\%$ (plus signs)

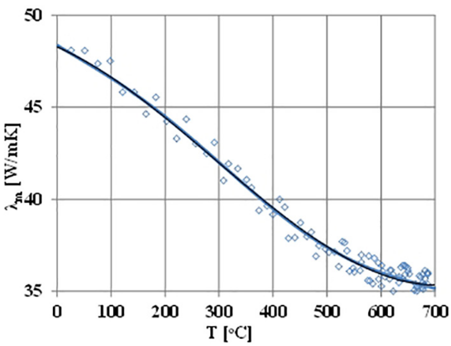

Figure 8.
Heat conduction coefficient λ_m for the metal as a function of temperature: undisturbed (solid line) and with disturbances $\pm 2.5\%$ (diamonds)

Figure 17 shows a graph of the matrix spectral radius for various thicknesses of the flat layer. The spectral radius $\rho = max\,((\lambda\,([a_{stabil}]^T\,[a_{stabil}]))^{0.5}) = \|\,a_{stabil}\,\|_2 = \sigma_1$ of the stability matrix, (26), is smaller than 1.0 up to a depth of approximately 4 mm from the outer surface. Figure 18 shows the spectral radius charts depending on the thickness of the flat layer related to the depth of insertion of the thermocouple (relative position is equal $\delta/(g+\delta)$).

Placing thermocouples at points where the spectral radius reaches the minimum is of great practical importance, as the smaller the value of the spectral radius, the greater the suppression of temperature measurement errors.

The solution of the Cauchy problem for a double-layer plate is a stable one, which was obtained by introducing the integral form of the energy balance equation in the area of ceramics. Therefore, this issue does not require regularization.

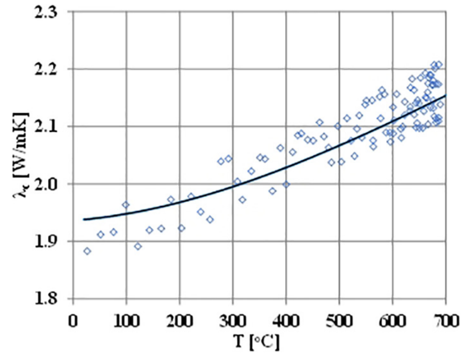

Figure 9.
Heat conduction coefficient λ_c for the ceramics as a function of temperature: undisturbed (solid line) and with disturbances $\pm 2.5\%$ (diamonds)

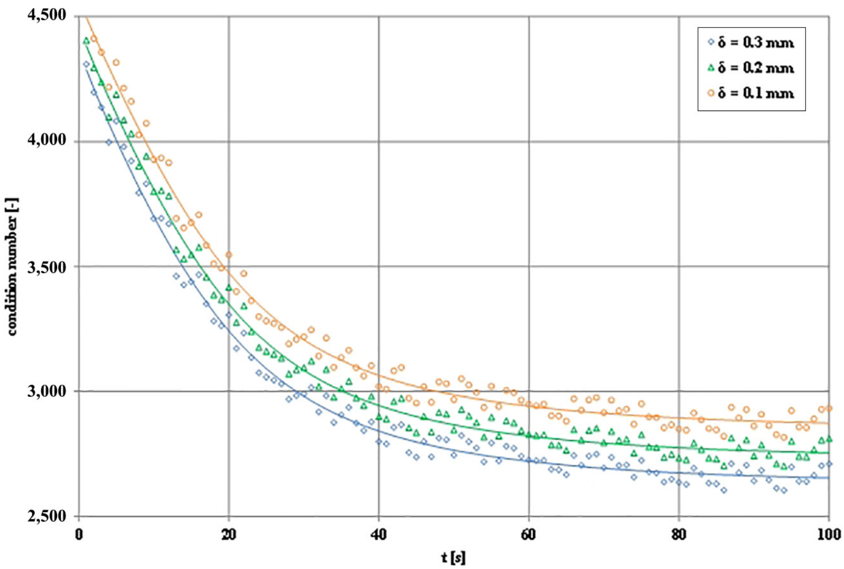

Figure 10.
Condition number of matrix $[a_{new}]$ (equation 24) for undisturbed temperature T_g (solid line) and disturbed temperature T_g (markers)

6. Conclusions

The non-linear inverse Cauchy problem solved in the present paper does not require regularization. This is due to the need to fulfil the energy balance equation (18) in the ceramics layer. The paper shows the influence of the depending on temperature heat conduction coefficient of the ceramics layer on the temperature on its outer surface. The lower the value of the heat transfer coefficient of the ceramics layer, the higher the temperature on the outer surface of the layer with the same temperature on the metal-ceramics interface. Therefore, it is important to choose the thickness of the ceramics layer δ in such a way that the temperature on the metal-ceramics interface does not reduce the mechanical properties of the metal.

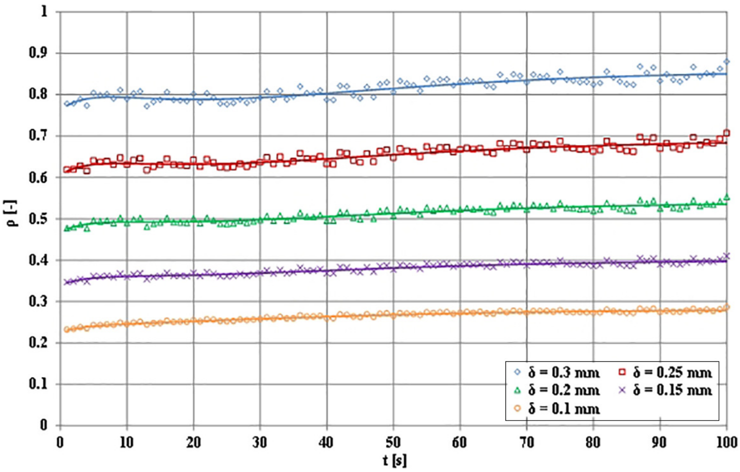

Figure 11. Spectral radius ρ of the stability matrix $[a_{stabil}]$ (formula (26)) for different ceramics thickness δ for undisturbed (solid line) and disturbed (markers) measurement of temperature T_g

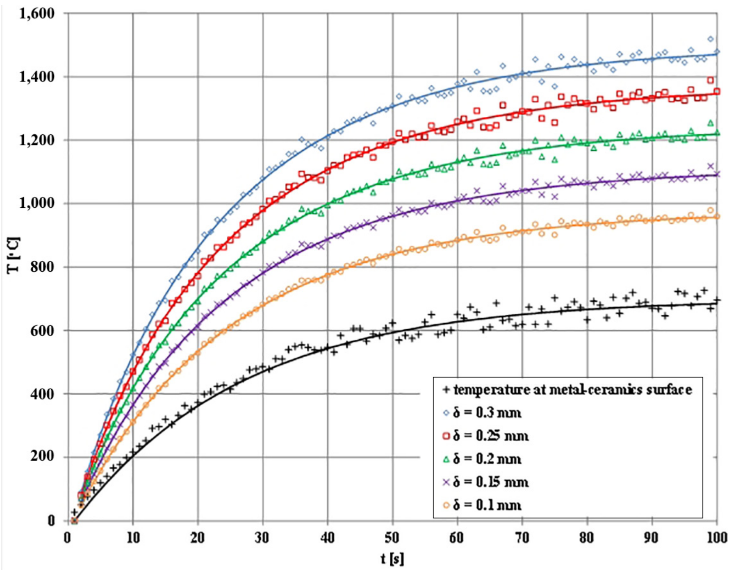

Figure 12. Influence of ceramics thickness on temperature on the outer ceramics surface for undisturbed (solid line) and disturbed (markers) measurement of temperature T_g

Due to the non-linearity of thermophysical parameters, the spectral radius ρ of the stability matrix, (26), is a function of time step. The thickness of the ceramics layer strongly affects the temperature on the metal-ceramics interface. The thicker the ceramics layer, the greater the difference between the temperature of the metal-ceramics interface and the outer surface of the ceramics. As the monitoring of heat load in a homogeneous material based on the measurement of temperature at the internal point is of great importance for the stable operation of the turbine, the answer to the question of how far (how deep) the thermocouple can be placed, so that the solution to the inverse problem is stable, is very important. This was tested for G20Mo5 steel characterized by variable

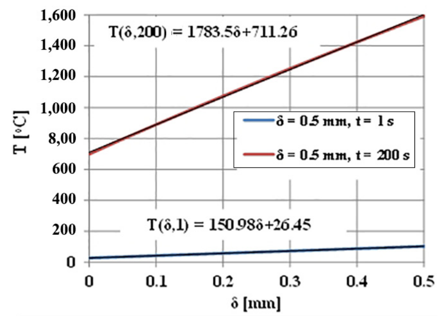

Figure 13.
Tempreture distribution in the ceramics thin layer

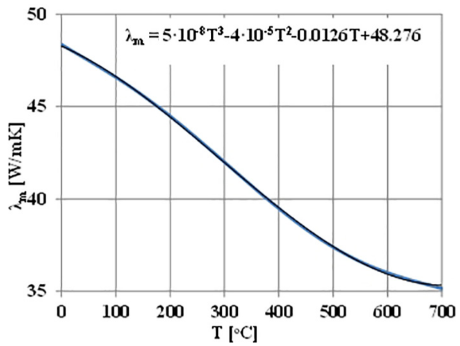

Figure 14.
Heat conduction coefficient λ_m for steel G20Mo5 as a function of temperature

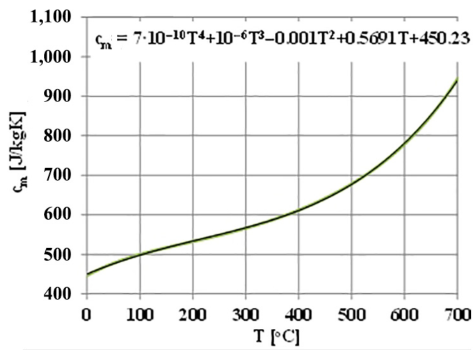

Figure 15.
Specific heat c_m for steel G20Mo5 as a function of temperature

thermophysical parameters as a function of temperature. The spectral radius of the stability matrix turned out to be less than 1.0 to a depth of about 4 mm from the outer surface. Placing thermocouples at points where the spectral radius reaches the minimum is of great practical importance, as the smaller the value of the spectral radius, the greater the suppression of temperature measurement errors.

The results presented in the paper are important in the selection of the thickness of the ceramics layer, which guarantees a decrease in temperature on the metal-ceramics interface to a safe value for

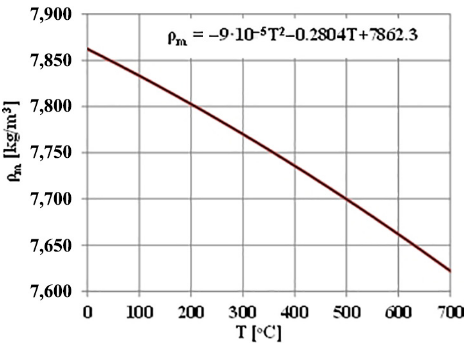

Figure 16. Density ρ_m for steel G20Mo5 as a function of temperature

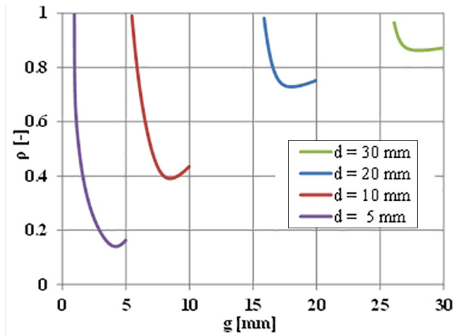

Figure 17. Stability intervalls for the stability matrix (26) for homogenous material (steel G20Mo5 in the first and second layer)

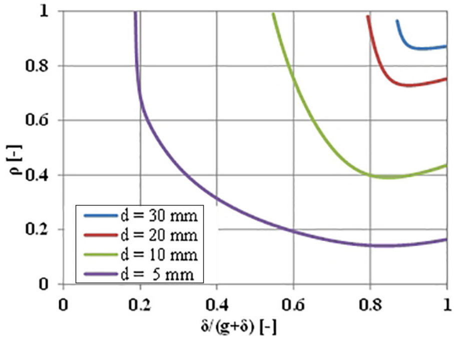

Figure 18. Stability intervalls for the stability matrix (26) for homogenous material (steel G20Mo5 in the first and second layer) and relative position of the thermocouple from the boundary

maintaining the mechanical properties of the metal. The next result of the analysis is the location of the thermocouple inside the area with homogeneous thermophysical properties to monitor thermal loads. This is of particular importance, among others, when starting up a thermal turbine, so as not to exceed the allowable thermal stresses.

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