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Choice of the regularization parameter for the Cauchy problem for the Laplace equation

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Abstract

Purpose – In this paper, the Cauchy-type problem for the Laplace equation was solved in the rectangular domain with the use of the Chebyshev polynomials. The purpose of this paper is to present an optimal choice of the regularization parameter for the inverse problem, which allows determining the stable distribution of temperature on one of the boundaries of the rectangle domain with the required accuracy.

Design/methodology/approach – The Cauchy-type problem is ill-posed numerically, therefore, it has been regularized with the use of the modified Tikhonov and Tikhonov–Philips regularization. The influence of the regularization parameter choice on the solution was investigated. To choose the regularization parameter, the Morozov principle, the minimum of energy integral criterion and the L-curve method were applied.

Findings – Numerical examples for the function with singularities outside the domain were solved in this paper. The values of results change significantly within the calculation domain. Next, results of the sought temperature distributions, obtained with the use of different methods of choosing the regularization parameter, were compared. Methods of choosing the regularization parameter were evaluated by the norm $N_{\max}$.

Practical implications – Calculation model described in this paper can be applied to determine temperature distribution on the boundary of the heated wall of, for instance, a boiler or a body of the turbine, that is, everywhere the temperature measurement is impossible to be performed on a part of the boundary.

Originality/value – The paper presents a new method for solving the inverse Cauchy problem with the use of the Chebyshev polynomials. The choice of the regularization parameter was analyzed to obtain a solution with the lowest possible sensitivity to input data disturbances.

Keywords Inverse Cauchy problem, L-curve, Minimum of energy integral criterion, Morozov principle, Regularization parameter, Laplace's equation, Regularization

Paper type Research paper

Nomenclature

a	= multinomial coefficient of the function of distribution of temperature $\tilde{T}(w)$;
c	= multinomial coefficient of the function of distribution of temperature $T(x, y)$;
$E(\alpha)$	= functional, energy integral;
J_α	= regularizing functional ($J_\alpha = J + \alpha^2 I$);
m	= number of Chebyshev nodes on the y-axis;
n	= number of Chebyshev nodes on the x-axis;
N_1–1	= degree of the polynomial describing unknown distribution of temperature on the Γ_1 boundary;

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$N_{\max}$ = norm;
q = heat flux density, W/m^{2}K;
T = temperature, K;
$\tilde{T}$ = temperature, function dependent on the Chebyshev node;
w = Chebyshev node;
W_i = Chebyshev polynomial of the first kind of i-th degree;
x, y = Cartesian coordinates;
$[x]_n$ = integer part of the division of number x by n; and
$x \bmod n$ = remainder of the division of number x by n.

Greek symbols
α = regularization parameter;
δ = error;
δ_M = error of measurement data (Morozov principle);
γ = multinomial coefficient, pertains to the sought temperature distribution on the boundary Γ_1;
Γ = boundary of the domain Ω, ($\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$); and
Ω = calculation domain.

Subscript
A = analytical solution;
c = calculated value;
m = measured value;
ran = randomly disturbed value; and
Γ_i = on the boundary Γ_i (for i = 1, 2, 3 and 4).

1. Introduction

Inverse problems are ill-posed in the Hadamard (1902) sense. It means that a slight disturbance to measurement data results in significant errors of the obtained results (Joachimiak and Ciałkowski, 2017, 2014, 2018; Nowak, 2017). Therefore, problems of such type need to be regularized. There are many methods used to regularize inverse problems. Among them, there is the Tikhonov regularization (Beck and Woodbury, 2016; Chen *et al.*, 2019; Djerrar *et al.*, 2017; Frąckowiak *et al.*, 2019a; Laneev, 2018; Marin, 2010, 2016; Niu *et al.*, 2014; Sun, 2016; Tikhonov and Arsenin, 1977; Yaparova, 2016), the Tikhonov–Philips regularization (Joachimiak *et al.*, 2019a), the discrete Fourier transform (Frąckowiak and Ciałkowski, 2018; Wróblewska *et al.*, 2015) and SVD algorithm (Hasanov and Mukanova, 2015). In her article, Cheruvu (2017) applied the wavelet regularization of Laplace's equation in the arbitrarily shaped domain. The solution to the Cauchy problem for the Laplace's equation was also sought with the use of the iterative Tikhonov-type method (Delvare and Cimetière, 2017). Han *et al.* (2011) in the article presented numerical tests concerning the solution to the Cauchy problem for Laplace's equation with the use of the energy regularization method. Obtained results were compared with the Tikhonov regularization for which the regularization parameter was chosen based on the Morozov principle. In the paper of Liu and Wang (2018), the Cauchy problem for the Laplace's equation was solved with the use of the method of fundamental solutions and the energy regularization technique to choose the source points. The Laplace's equation was also solved with the use of iterative algorithms (Frąckowiak *et al.*, 2015a, 2015b), of the Trefftz method (Ciałkowski and Frąckowiak, 2002; Ciałkowski and Grysa, 2010; Grysa *et al.*, 2012; Hożejowski, 2016; Lin *et al.*, 2018), of the method of fundamental solution (Kołodziej and Mierzwiczak, 2008; Mierzwiczak *et al.*, 2015; Mierzwiczak and Kołodziej, 2011) and of the collocation method (Joachimiak *et al.*, 2016). In many cases, the regularization of the inverse problem concerns the problem of choosing the regularization parameter. The regularization parameter can be chosen based on the Morozov principle (Chen *et al.*, 2019; Han *et al.*, 2011; Joachimiak *et al.*, 2019a; Marin, 2016; Morozov, 1984; Sun, 2016) or using the L-curve method (Jin and Zheng, 2006; Marin and Munteanu, 2010; Marin, 2005). In the study of Marin (2011), the optimal regularization parameter was sought based on the generalized cross-validation criterion. Currently, research work focuses on finding new methods of regularization (Cheng and Feng, 2014; Zhuang and Chen, 2017) and on the modification of already known and used methods (Yang *et al.*, 2015; Zheng and Zhang, 2018). Because of a wide application of inverse problems in engineering problems, such as the cooling of the blades in gas turbines (Frąckowiak *et al.*, 2017; Frąckowiak *et al.*, 2019b; Frąckowiak *et al.*, 2011), analysis of the boiling heat transfer in minichannels (Hożejowska *et al.*, 2009; Maciejewska and Piasecka, 2017), analysis of thermal and thermo-chemical treatment (Joachimiak *et al.*, 2019b) or monitoring of power boilers operation (Taler *et al.*, 2016, 2017), developing methods for regularization of these problems and investigating the process of choosing the regularization parameter are very significant.

In this article, the solution to the Cauchy problem for the Laplace's equation was investigated with the use of the Chebyshev polynomials. To regularize the solution to the inverse problem, the modification of the Tikhonov and of the Tikhonov–Philips regularizations, described in the article (Joachimiak *et al.*, 2019a) was applied. The choice of the regularization parameter was made based on the Morozov principle, the minimum energy integral criterion and the L-curve method.

2. Calculation model

In many technical applications, it is impossible to measure temperature on the boundary of the heated component of the device or machine, such as the combustion chamber, the inner side of the body of a turbine or a boiler. Then, the distribution of temperature can be determined by finding the solution to the inverse problem. Based on the distribution of temperature on the part of the boundary $\left(T_{\Gamma_2}, T_{\Gamma_3}, T_{\Gamma_4}, \text{fig. 1}\right)$ and, additionally, knowing the heat flux density on the part of the boundary $\left(q_{\Gamma_3}, \text{fig. 1}\right)$ one can determine the distribution of temperature on the boundary, where it is impossible to measure this temperature $\left(T_{\Gamma_1}, \text{fig. 1}\right)$. Such a posed problem is the Cauchy problem, particularly sensitive to errors in measurement and in the calculation. In the stationary thermal field, the heat equation is reduced to the Laplace's equation (for the non-linear case, the Kirchhoff's substitution transforms the heat equation into the Laplace's equation).

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \tag{1}$$

Laplace's equation is solved in the domain $\Omega = \{(x, y) \in \mathbb{R}^2 \colon -1 \le x \le 1, -1 \le y \le 1\}$ with the following boundary conditions (Figure 1):

$$T(x, y = 1) = T_{\Gamma_2}(x) \text{ for } -1 \le x \le 1 \tag{2}$$

$$T(x = -1, y) = T_{\Gamma_3}(y) \text{ for } -1 \le y \le 1 \tag{3}$$

$$\frac{\partial T(x = -1, y)}{\partial n} = q_{\Gamma_3}(y) \text{ for } -1 \le y \le 1 \tag{4}$$

$$T(x, y = -1) = T_{\Gamma_4}(x) \text{ for } -1 \le x \le 1 \tag{5}$$

It was assumed that the solution can be noted as the linear combination of the Chebyshev polynomials (Paszkowski, 1975).

$$T(x_i, y_j) = \sum_{p=0}^{n-1} \sum_{q=0}^{m-1} c_{pq} W_p(x_i) W_q(y_j) \tag{6}$$

To solve the Cauchy problem, the collocation method was used. It was assumed that there are n points along the x-axis and m points along the y-axis (including points on the boundary). Collocation points being inside the interval (−1, 1) are the Chebyshev nodes (Paszkowski, 1975). Nodes were renumbered, which enables the temperature function to be noted in the following equation (7):

$$\tilde{T}(w_l) = \sum_{k=1}^{mn} a_k W_{[k-1]_m}\left(x_{(l-1)\bmod n+1}\right) W_{(k-1)\bmod m}\left(y_{[l-1]_n+1}\right) \tag{7}$$

where the coefficients a_k (k = 1, 2, . . ., mn) are unknown. Sought temperature distribution on the boundary Γ_1 was assumed as the linear combination of the Chebyshev polynomials (Paszkowski, 1975).

$$T_{\Gamma_1}(y) = \sum_{h=1}^{N_1} \gamma_h W_{h-1}(y) \tag{8}$$

Coefficients a_k (k = 1, 2, . . ., mn) are expressed by the values of coefficients γ_i (i = 1, 2, . . ., N_1). Hence, the determination of the temperature distribution is reduced to the determination of coefficients γ_i. To do so, the functional of the following form was minimized.

$$J = \int_{\Gamma_2} \left(T_{c,\Gamma_2} - T_{m,\Gamma_2}\right)^2 d\Gamma_2 + \int_{\Gamma_3} \left(T_{c,\Gamma_3} - T_{m,\Gamma_3}\right)^2 d\Gamma_3 + \int_{\Gamma_3} \left(\frac{\partial T_{c,\Gamma_3}}{\partial n} - q_{m,\Gamma_3}\right)^2 d\Gamma_3 + \int_{\Gamma_4} \left(T_{c,\Gamma_4} - T_{m,\Gamma_4}\right)^2 d\Gamma_4 \tag{9}$$

where c in subscript denotes the calculated value, while m denotes the measured value. The integral in equation (9) on the boundary Γ_2 can be noted in the following equation (10):

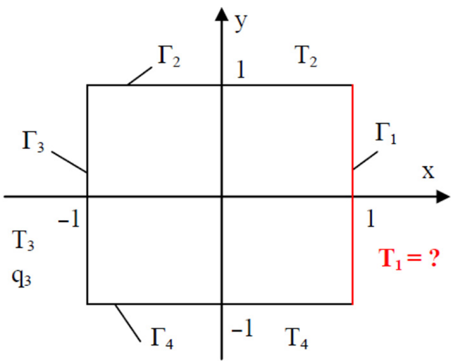

Figure 1.
Calculation domain

$$J_{\Gamma_2} = \int_{\Gamma_2} \left(T_{c,\Gamma_2} - T_{m,\Gamma_2}\right)^2 d\Gamma_2 = \sum_{i=1}^{n} \int_{\Gamma_{2i}} \left(T_{c,\Gamma_{2i}} - T_{m,\Gamma_{2i}}\right)^2 d\Gamma_{2i} \tag{10}$$

Applying numerical integration we have:

$$J_{\Gamma_2} = \sum_{i=1}^{n} \Delta x_i (T_c(x_i, 1) - T_m(x_i, 1))^2 = \sum_{i=1}^{n} \left(\sqrt{\Delta x_i}(T_c(x_i, 1) - T_m(x_i, 1))\right)^2 \tag{11}$$

where $\Delta x_1 = \frac{x_2 - x_1}{2}$, $\Delta x_i = \frac{x_{i+1} - x_{i-1}}{2}$ for $i = 2, 3, \ldots, n-1$ and $\Delta x_n = \frac{x_n - x_{n-1}}{2}$. Having inserted the equation (7) into the equation (11), we obtained:

$$J_{\Gamma_2} = \sum_{i=1}^{n} \left(\sqrt{\Delta x_i}\left(\sum_{k=1}^{mn} a_k W_{[k-1]_m}(x_i) W_{(k-1)\mathrm{mod}m}(1) - T_m(x_i, 1)\right)\right)^2 \tag{12}$$

Solving the direct problem, where the temperature on boundaries Γ_1, Γ_2, Γ_3 and Γ_4 was known, was reduced to solving the matrix equation

$$Ax = b \tag{13}$$

what was described in detail in the paper (Joachimiak *et al.*, 2019a). Based on the solution to the direct problem, constants a_k [Equation (12)] are of the following equation (14):

$$a_k = F_k + \sum_{h=1}^{N_1} \gamma_h H_{h,k} \tag{14}$$

where $H_{h,k} = \sum_{j=2}^{m-1} \tilde{A}_{k,jn} W_{h-1}(y_j)$, while $\tilde{A}_{k,jn}$ ($k = 1, 2, \ldots, mn$; $j = 2, 3, \ldots, m-1$) are elements of the matrix A^{-1}. After substituting equation (14) into equation (12) we obtained:

$$\begin{aligned} J_{\Gamma_2} &= \sum_{i=1}^{n} \left(\sqrt{\Delta x_i}\left(\sum_{k=1}^{mn}\left(F_k + \sum_{h=1}^{N_1} \gamma_h H_{h,k}\right) W_{[k-1]_m}(x_i) W_{(k-1)\mathrm{mod}m}(1) - T_m(x_i, 1)\right)\right)^2 \\ &= \sum_{i=1}^{n} \left(\sum_{h=1}^{N_1} \gamma_h \sum_{k=1}^{mn} \sqrt{\Delta x_i} H_{h,k} W_{[k-1]_m}(x_i) W_{(k-1)\mathrm{mod}m}(1)\right. \\ &\quad \left. + \sum_{k=1}^{mn} \sqrt{\Delta x_i} F_k W_{[k-1]_m}(x_i) W_{(k-1)\mathrm{mod}m}(1) - T_m(x_i, 1)\right)^2 \\ &= \sum_{i=1}^{n} \left(\sum_{h=1}^{N_1} \gamma_h C_1(i, h) - D_1(i)\right)^2 \end{aligned} \tag{15}$$

We would like the integral J_{Γ_2} to have a value equal to zero or as close to zero as possible, hence, we equate the squared expression [Equation (15)] to zero. Hence, we have that:

$$\underset{i=1,2,\ldots,n}{\forall} \sum_{h=1}^{N_1} \gamma_h C_1(i,h) = D_1(i) \tag{16}$$

We obtain n linear equations of the following equation (17):

$$\{C_1(i,h)\}\{\gamma_h\} = D_1(i) \tag{17}$$

for $i = 1, 2, \ldots, n$ and $h = 1, 2, \ldots, N_1$. It can be reduced to the matrix equation.

$$[C_{1,n}]\{\gamma\} = \{D_{1,n}\} \tag{18}$$

Similarly, for other integrals [Equation (9)] we obtained:

$$J_{\Gamma_3} = \int_{\Gamma_3} \left(T_{c,\Gamma_3} - T_{m,\Gamma_3}\right)^2 d\Gamma_3 = \sum_{i=1}^{m} \left(\sum_{h=1}^{N_1} \gamma_h C_2(i,h) - D_2(i) \right)^2 \tag{19}$$

$$J_{q,\Gamma_3} = \int_{\Gamma_3} \left(\frac{\partial T_{c,\Gamma_3}}{\partial n} - q_{m,\Gamma_3} \right)^2 d\Gamma_3 = \sum_{i=1}^{m} \left(\sum_{h=1}^{N_1} \gamma_h C_3(i,h) - D_3(i) \right)^2 \tag{20}$$

$$J_{\Gamma_4} = \int_{\Gamma_4} \left(T_{c,\Gamma_4} - T_{m,\Gamma_4}\right)^2 d\Gamma_4 = \sum_{i=1}^{n} \left(\sum_{h=1}^{N_1} \gamma_h C_4(i,h) - D_4(i) \right)^2 \tag{21}$$

After J_{Γ_3}, J_{q,Γ_3} and J_{Γ_4} had been equated to zero, equations of the following forms were obtained:

$$\underset{i=1,2,\ldots,m}{\forall} \sum_{h=1}^{N_1} \gamma_h C_2(i,h) = D_2(i) \tag{22}$$

$$\underset{i=1,2,\ldots,m}{\forall} \sum_{h=1}^{N_1} \gamma_h C_3(i,h) = D_3(i) \tag{23}$$

$$\underset{i=1,2,\ldots,n}{\forall} \sum_{h=1}^{N_1} \gamma_h C_4(i,h) = D_4(i) \tag{24}$$

Based on equations (16) and (22)-(24), an oversized system of linear equations was obtained as the matrix equation, which would be solved with the use of the SVD algorithm:

$$\begin{bmatrix} [C_{1,n}] \\ [C_{2,m}] \\ [C_{3,m}] \\ [C_{4,n}] \end{bmatrix} \{\gamma\} = \begin{Bmatrix} \{D_{1,n}\} \\ \{D_{2,m}\} \\ \{D_{3,m}\} \\ \{D_{4,n}\} \end{Bmatrix} \tag{25}$$

what can be noted in the shorter form:

$$[BM]\{\gamma\} = \{BW\} \tag{26}$$

Because of a great sensitivity of results to disturbances to measurement data, the Cauchy problem was regularized. The regularizing functional of the following form was assumed:

$$J_\alpha = J(\gamma) + \alpha^2 I(\gamma) = \left\| [BM]\{\gamma\} - \{BW\} \right\|^2 + \alpha^2 \int\limits_{\Gamma_1} \left((\tilde{T})^2 + \left(\frac{\partial \tilde{T}}{\partial y} \right)^2 \right) d\Gamma_1 \tag{27}$$

Regularization term can be noted as the sum of integrals.

$$\alpha^2 I(\gamma) = \alpha^2 \int\limits_{\Gamma_1} \left((\tilde{T})^2 + \left(\frac{\partial \tilde{T}}{\partial y} \right)^2 \right) d\Gamma_1 = \alpha^2 \sum_{i=1}^{m-1} \int\limits_{\Gamma_{1i}} \left((\tilde{T})^2 + \left(\frac{\partial \tilde{T}}{\partial y} \right)^2 \right) d\Gamma_{1i} \tag{28}$$

where $\Gamma_1 = \bigcup\limits_{i=1}^{m-1} \Gamma_{1i}$. Performing numerical integration using the trapezoidal rule, we obtained:

$$\alpha^2 I(\gamma) = \sum_{i=1}^{m-1} \alpha^2 \frac{y_{i+1} - y_i}{2} \left[\left(\tilde{T}(1, y_{i+1}) \right)^2 + \left(\frac{\partial \tilde{T}(1, y_{i+1})}{\partial y} \right)^2 + \left(\tilde{T}(1, y_i) \right)^2 + \left(\frac{\partial \tilde{T}(1, y_i)}{\partial y} \right)^2 \right] \tag{29}$$

On the boundary Γ_1 we have:

$$\begin{aligned} \tilde{T}(1, y_i) &= \sum_{k=1}^{mn} a_k W_{[k-1]_m}(1) W_{(k-1)\mathrm{mod}m}(y_i) = \sum_{k=1}^{mn} \left(F_k + \sum_{h=1}^{N_1} \gamma_h H_{h,k} \right) W_{[k-1]_m}(1) W_{(k-1)\mathrm{mod}m}(y_i) = \\ &= \sum_{k=1}^{mn} F_k W_{[k-1]_m}(1) W_{(k-1)\mathrm{mod}m}(y_i) + \sum_{h=1}^{N_1} \gamma_h \sum_{k=1}^{mn} H_{h,k} W_{[k-1]_m}(1) W_{(k-1)\mathrm{mod}m}(y_i) = \\ &= A_1(i) + \sum_{h=1}^{N_1} \gamma_h A_2(i, h) \end{aligned} \tag{30}$$

$$\begin{aligned} \frac{\partial \tilde{T}(1, y_i)}{\partial y} &= \sum_{k=1}^{mn} F_k W_{[k-1]_m}(1) W'_{(k-1)\mathrm{mod}m}(y_i) + \sum_{h=1}^{N_1} \gamma_h \sum_{k=1}^{mn} H_{h,k} W_{[k-1]_m}(1) W'_{(k-1)\mathrm{mod}m}(y_i) = \\ &= A_3(i) + \sum_{h=1}^{N_1} \gamma_h A_4(i, h) \end{aligned} \tag{31}$$

Hence,

$$\alpha^2 I(\gamma) = \sum_{i=1}^{m-1}\left\{\alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_1(i+1)+\sum_{h=1}^{N_1}\gamma_h A_2(i+1,h)\right)\right\}^2$$
$$+\sum_{i=1}^{m-1}\left\{\alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_3(i+1)+\sum_{h=1}^{N_1}\gamma_h A_4(i+1,h)\right)\right\}^2$$
$$+\sum_{i=1}^{m-1}\left\{\alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_1(i)+\sum_{h=1}^{N_1}\gamma_h A_2(i,h)\right)\right\}^2$$
$$+\sum_{i=1}^{m-1}\left\{\alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_3(i)+\sum_{h=1}^{N_1}\gamma_h A_4(i,h)\right)\right\}^2 \quad (32)$$

Each of the components [Equation (32)] was equated to zero. The equation of the following form was obtained:

$$\underset{i=1,2,\ldots,m-1}{\forall}\ \alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_1(i+1)+\sum_{h=1}^{N_1}\gamma_h A_2(i+1,h)\right)=0 \quad (33)$$

$$\underset{i=1,2,\ldots,m-1}{\forall}\ \alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_3(i+1)+\sum_{h=1}^{N_1}\gamma_h A_4(i+1,h)\right)=0 \quad (34)$$

$$\underset{i=1,2,\ldots,m-1}{\forall}\ \alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_1(i)+\sum_{h=1}^{N_1}\gamma_h A_2(i,h)\right)=0 \quad (35)$$

$$\underset{i=1,2,\ldots,m-1}{\forall}\ \alpha\sqrt{\frac{y_{i+1}-y_i}{2}}\left(A_3(i)+\sum_{h=1}^{N_1}\gamma_h A_4(i,h)\right)=0 \quad (36)$$

Equations (33)-(36) can be reduced to the following system of equations.

$$\alpha[CM]\{\gamma\} = -\alpha\{CW\} \quad (37)$$

where α is the regularization parameter. When the equation (26) and regularization [Equation (37)] are included, the following system of equations is obtained:

$$\begin{bmatrix}[BM]\\ \alpha[CM]\end{bmatrix}\{\gamma\} = \begin{Bmatrix}\{BW\}\\ -\alpha\{CW\}\end{Bmatrix} \quad (38)$$

The solution to the system of equations (38) was sought in the least-squares sense with the use of the SVD algorithm.

3. Choice of the regularization parameter

To determine unknown regularization parameter α, the Morozov principle, the minimum of energy integral criterion and L-curve method were applied. For the solution obtained with the use of the Morozov principle, the mean (Morozov_A) and the maximal (Morozov_B) errors of the heat flux density δ_M on the boundary Γ_3 were evaluated. Interval halving method was used to determine zero of the function $F_M(\alpha)$ defined by the following equation (39):

$$\left\| [BM]\{\gamma\} - \{BW\} \right\|^2 - \delta_M^2 = F_M \tag{39}$$

Unknown regularization parameter α was also sought based on the minimization of the functional (energy integral) of the following equation (40):

$$E(\alpha) = \int_{\Omega} (\nabla T(\alpha))^2 d\Omega, \; T(\alpha) \in C^2(\Omega) \tag{40}$$

where $(\nabla T(\alpha))^2 = \left(\frac{\partial T(\alpha)}{\partial x}\right)^2 + \left(\frac{\partial T(\alpha)}{\partial y}\right)^2$. The minimum of the energy integral corresponds to satisfying the Laplace's equation (with respective boundary conditions), which is discussed in the paper (Gelfand and Fomin, 1979). Therefore, we choose the parameter α for which $\min_{\alpha} E(\alpha)$ occurs, i.e. the derivative $E' = \frac{dE}{d\alpha}$ reverses the sign. Domain Ω was divided into rectangular domains with the use of equidistant nodes and next into domains being right-angled triangles. The integral value was calculated with the use of the finite element method. Value ∇T was determined based on the form of the solution equation (6). Values γ_h were obtained by solving the equation (38).

On the basis of the solution of the equation (38), the L-curve was drawn as the correlation between $\|[BM]\{\gamma\} - \{BW\}\|$ on the x-axis and $\|[CM]\{\gamma\} - \{CW\}\|$ on the y-axis. We sought for the regularization parameter α with which corresponded the point of the L-curve locating on the curvature of this line. To evaluate the choice of the regularization parameter α, the following norm was defined:

$$N_{\max} = \frac{\max\left| T_{\Gamma 1\_C} - T_{\Gamma 1\_A} \right|}{\max\left| T_{\Gamma 1\_A} \right|} \tag{41}$$

4. Numerical examples

Calculations were made in the domain Ω for the function.

$$f_1 = \ln\left((x-a)^2 + (y-b)^2\right), q_{1,\Gamma_3} = \frac{2(-1-a)}{(-1-a)^2 + (y-b)^2} \tag{42}$$

and

$$f_2 = \mathrm{Re}\left(\frac{1}{z-(a+bi)}\right) = \frac{x-a}{(x-a)^2+(y-b)^2}, q_{2,\Gamma_3} = \frac{-(-1-a)^2 + (y-b)^2}{\left[(-1-a)^2 + (y-b)^2\right]^2} \tag{43}$$

We assumed such values of constants a and b that singularities would be outside the calculation domain Ω and that the values of gradients would change significantly within this domain (a = 1.3, b = 1.3, b = 1.1). Values of the norm $N_{\max}$ [Equation (41)], not including the regularization [Equation (26)], without disturbance (δ_{ran} = 0) and with random disturbance to the heat flux density up to 0.01q (δ_{ran} = 0.01) and to 0.02q (δ_{ran} = 0.02) are summarized in Table I. Disturbance q is an additive function with the uniform distribution. A slight disturbance to measurement data results in a significant error of the sought temperature on the boundary Γ_1. Hence, it is necessary to regularize the inverse problem [Equation (38)] and to choose the regularization parameter α properly.

4.1 Example 1

Calculations were made for the function f_1 [Equation (42)]. Heat flux density was disturbed randomly to 0.02q (δ_{ran} = 0.02). Regularization parameter α was chosen with the use of the Morozov principle, the minimum of energy integral criterion and the L-curve method.

Figure 2 presents the course of the energy integral and its derivative depending on the parameter α. To solve the Cauchy problem, the authors applied such value of the regularization parameter α for which the energy integral $E(\alpha)$ took the minimal value, which meant that the derivative of the energy integral $E' = \frac{dE}{d\alpha}$ reversed the sign. The value of α was 5.13 $\times$ 10^{-4} (Table II). The Cauchy problem was also regularized for

Table I. Values of the norm N_{max} for calculations without regularization (α = 0), without disturbance (δ_{ran} = 0) and with random disturbance to the heat flux density up to 0.01q (δ_{ran} = 0.01) and 0.02q (δ_{ran} = 0.02)

Error of heat flux density disturbed randomly	$N_{\max}$		
		f_1	f_2
δ_{ran} = 0	$2.78{\cdot}10^{-2}$	0.40047	
δ_{ran} = 0.01	21978535	2814929	
δ_{ran} = 0.02	43957070	5629859	

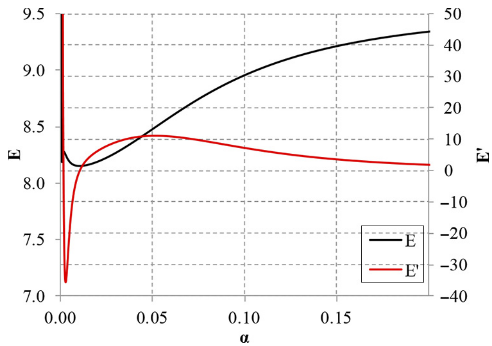

Figure 2. Energy integral (E) and the derivative of the energy integral (E') depending on the value of the regularization parameter α (function f_1)

the regularization parameter α amounting to 4.64×10^{-1}, which was determined based on the L-curve course (Figure 3).

To choose the regularization parameter with the Morozov principle, the values of mean and maximal error δ_M were evaluated for the heat flux on the boundary Γ_3. The respective values were obtained: 0.008 and 0.02 (Table II). Next, zero of the function F_M (α) was calculated as per the equation (39). The respective values of the regularization parameter were obtained: 1.98×10^{-6} and 4.0799×10^{-2} (Table II).

The lowest value of the norm N_{max} amounting to 6.18×10^{-2} (Table II) for the function f_1 was obtained for the case of choosing the regularization parameter with the use of the Morozov principle for the maximal error of the heat flux δ_M (Morozov_B). This criterion brought satisfying results, as did the choice of the regularization parameter made with the use of the minimum energy integral criterion ($N_{max} = 9.796 \times 10^{-2}$). When the L-curve method was used, the obtained results were considerably worse ($N_{max} = 2.22 \times 10^{-1}$). For the Morozov principle, for the mean error δ_M of the heat flux (Morozov_A), the highest value of the norm N_{max} amounting to 50.42 was obtained. Distributions of temperature on the boundary Γ_1 resulting from the analytical solution (AS) and from the solution of the Cauchy problem are presented in Figure 4.

4.2 Example 2

Calculations were made for the function f_2 [Equation (43)]. Heat flux density was disturbed randomly to $0.02q$ ($\delta_{ran} = 0.02$). The best results were obtained for the choice of the regularization parameter made with the use of the minimum energy integral criterion

Method of the choice of the regularization parameter	δ_M	α	N_{max}
Morozov_A	0.008	1.98×10^{-6}	50.42
Morozov_B	0.02	4.0799×10^{-2}	6.18×10^{-2}
E	–	5.13×10^{-4}	9.796×10^{-2}
L-curve	–	4.64×10^{-1}	2.22×10^{-1}

Table II. Values of the measurement data error δ_M, of the regularization parameter α and of the norm N_{max} for the choice of the regularization parameter α made using the Morozov principle (Morozov_A and Morozov_B), the minimum of energy integral criterion (E) and the L-curve method (L-curve) for the function f_1

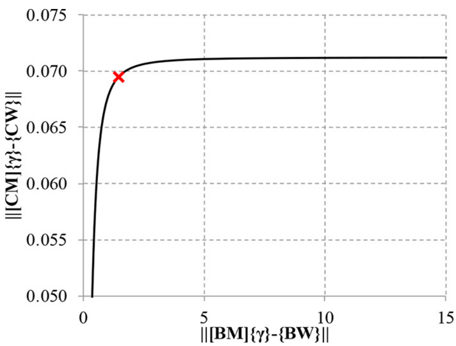

Figure 3. L-curve with the point for which the regularization parameter α was chosen (function f_1)

(N_{max} = 4.77 × 10^{-2}), and the worse results were obtained for the L-curve method (N_{max} = 0.361). Values of the regularization parameter and of the norm N_{max}, being the measure of the quality of the parameter α choice, for the function f_2 are summarized in Table III. Distributions of temperature on the boundary Γ_1 for the AS and for the solution to the Cauchy problem with regularization are presented in Figure 5. Distribution of temperature on the boundary Γ_1 obtained with the use of the minimum of energy integral criterion slightly diverges from the AS.

To examine thoroughly the criterion for the regularization parameter selection with the use of the minimum of energy integral, calculations were performer also for the following functions:

$$f_3 = \cos x \cosh y + \sin x \sinh y \tag{44}$$

$$f_4 = e^x \sin y \tag{45}$$

$$f_5 = x^3 - 3xy^2 + e^{2y}\sin 2x - e^x \cos y \tag{46}$$

$$f_6 = e^{x^2 - y^2}\sin 2xy \tag{47}$$

which were chosen based on publications (Liu *et al.*, 2018; Conde Mones *et al.*, 2017; Fu *et al.*, 2013; Sun, 2017). Values of the regularization parameter and of the norm N_{max} for functions $f_3 - f_6$ are summarized in Table IV. For the function f_6 and the disturbance to the heat flux density δ_{ran} = 0.02 and δ_{ran} = 0.05 the minimum of energy integral was not achieved. Distributions of temperature on the boundary Γ_1 being sought are presented in Figure 6. For

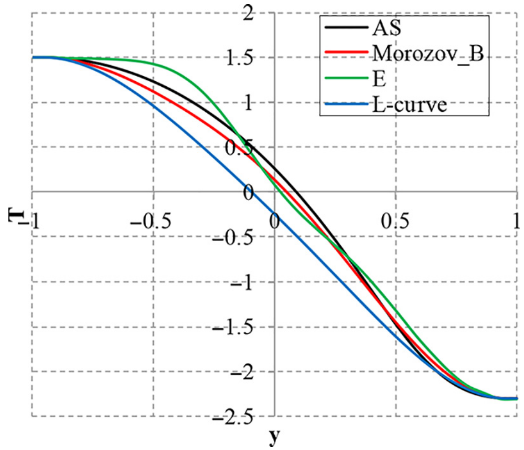

Figure 4.
Distribution of temperature on the boundary Γ_1 obtained from the AS and form the Cauchy problem when the regularization parameter α was chosen with the use of the Morozov principle (Morozov_B), the minimum of energy integral criterion (E) and the L-curve method (L-curve) for the function f_1

functions $f_3 - f_5$ the disturbance $\delta_{ran} = 0.05$ was taken into account, and for the function f_6 it was $\delta_{ran} = 0.01$.

5. Conclusion

This paper presents the solution of the Cauchy problem for Laplace's equation. Obtained distributions of temperature on the boundary Γ_1 were analyzed in terms of the dependence on the method for choosing the regularization parameter. The best results were obtained for the choice of the regularization parameter made with the use of the minimum of energy integral criterion and the Morozov principle (δ_M is the maximal error for the heat flux on the boundary Γ_3). The advantage of the application of the minimum energy integral criterion is a unique determination of the regularization parameter α for which $E(\alpha)$ has minimal value. Regularization made with the use of the minimum energy integral criterion gives satisfying results. However, its disadvantage is the fact that not for all calculation examples the minimum energy integral was determined. For the Morozov principle, the obtained

Table III. Values of the measurement data error δ_M, of the regularization parameter α and of the norm $N_{\max}$ for the choice of the regularization parameter α made using the Morozov principle (Morozov_A and Morozov_B), the minimum of energy integral criterion (E) and of the L-curve method (L-curve) for the function f_2

Method of the choice of the regularization parameter	δ_M	α	$N_{\max}$
Morozov_A	0.002	1.84×10^{-3}	0.115
Morozov_B	0.004	6.702×10^{-3}	0.174
E	–	3.37×10^{-4}	4.77×10^{-2}
L-curve	–	3.998×10^{-1}	0.361

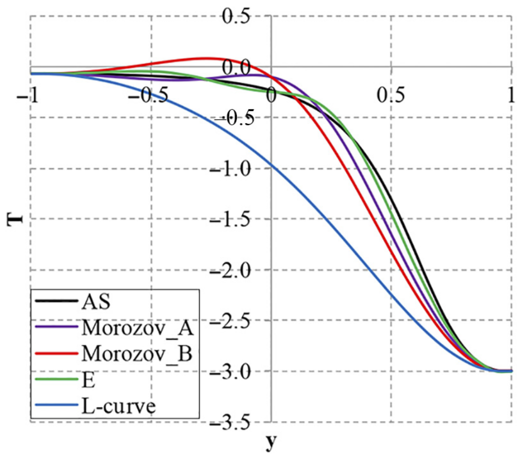

Figure 5. Distribution of temperature on the boundary Γ_1 obtained from the AS and form the solution to the Cauchy problem when the regularization parameter α was chosen with the use of the Morozov principle (Morozov_A and Morozov_B), the minimum of energy integral criterion (E) and of the L-curve method (L-curve) for the function f_2

results (distribution of temperature on the boundary Γ_1) depend on calculation or evaluation of the value of the heat flux δ_M error, what is a disadvantage of this method. Inexact evaluation of the measurement data error can result in obtaining the distribution of temperature on the boundary Γ_1, which is subject to great uncertainty. The choice of the parameter α made with the use of the L-curve method gave the worst results. Smooth L-curve course was obtained, what was related to the problem with the unique determination of the regularization parameter α using this method.

Table IV.
Values of the regularization parameter and of the norm $N_{\max}$ for functions $f_3 - f_6$ with the disturbance to the heat flux density δ_{ran} from 0.01 to 0.05

Function	$\delta_{ran} = 0.01$		$\delta_{ran} = 0.02$		$\delta_{ran} = 0.05$	
	α	N_{max}	α	N_{max}	α	N_{max}
f_3	2.98×10^{-2}	2.38×10^{-3}	2.89×10^{-2}	4.49×10^{-3}	2.803×10^{-2}	1.107×10^{-2}
f_4	4.99×10^{-4}	3.46×10^{-2}	1.0×10^{-3}	3.98×10^{-2}	1.0×10^{-3}	6.406×10^{-2}
f_5	4.99×10^{-4}	3.46×10^{-2}	1.0×10^{-3}	3.98×10^{-2}	1.0×10^{-2}	2.16×10^{-1}
f_6	1.0×10^{-3}	2.33×10^{-1}	No minimum		No minimum	

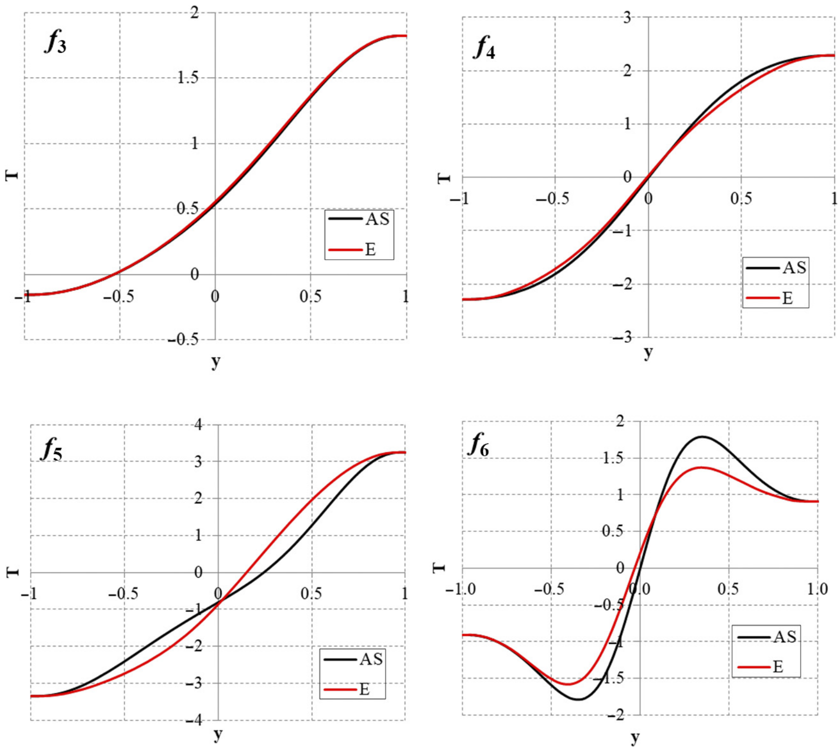

Figure 6.
Distribution of temperature on the boundary Γ_1 obtained from the AS and form the solution to the Cauchy problem when the regularization parameter α was chosen with the use of the minimum of energy integral criterion (E) for functions $f_2 - f_6$

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